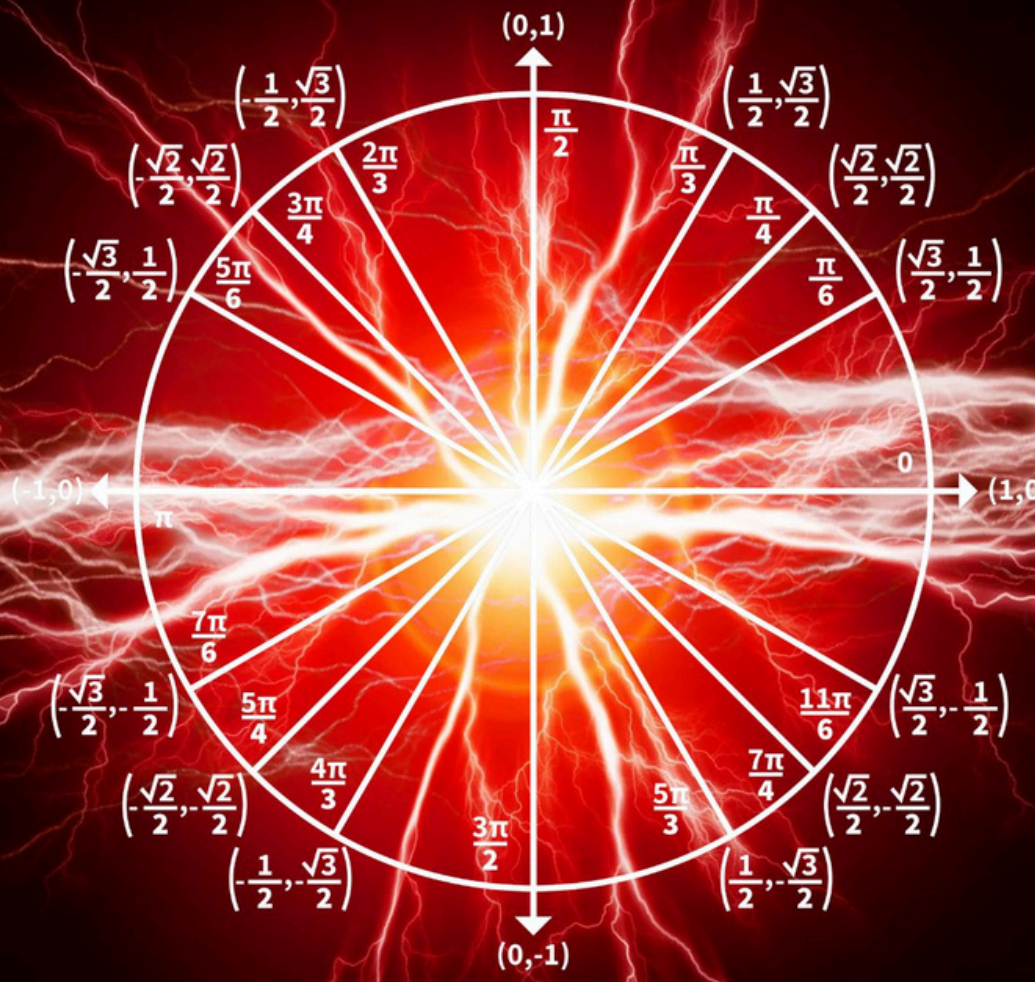
MATH MONTHLY

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B.S. Mathematics (pure)

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PLEM Academy (Physics • Language • Engineering • Mathematics)

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This publication is an educational companion and study guide. It is not a substitute for formal education, accredited coursework, classroom instruction, tutoring, engineering judgment, scientific research, professional certification, or independent study. Readers are responsible for verifying all information with their instructors, official course materials, and authoritative academic references.

The Ultimate Crash Course for PLEM Majors is designed to teach students how to learn from textbooks, not to replace them. Throughout this series, the author discusses concepts, problem types, and solution methods commonly found in widely used college mathematics, physics, engineering, and related STEM textbooks. Students are encouraged to own, read, and properly cite the textbooks assigned by their instructors.

The purpose of this series is to demonstrate effective study methods, textbook navigation, note-taking, scientific communication, mathematical writing, and proper academic citation. Readers are encouraged to develop professional problem-solving skills by writing complete mathematical essays that include a descriptive title, introduction, logical body, supporting references, conclusions, and properly documented sources. The goal is to move beyond simply obtaining answers and instead develop the ability to communicate mathematical and scientific ideas professionally.

This series also promotes academic integrity. The author does not encourage plagiarism, unauthorized collaboration, contract cheating, or any other violation of institutional academic policies. Modern technologies, including artificial intelligence, should be used ethically as tools for organization, formatting, research, and communication—not as replacements for genuine understanding, reasoning, or original work.

This work was written in response to the growing decline in study habits, textbook literacy, and independent critical thinking among many college students. The author encourages students to rebuild strong academic foundations through disciplined reading, careful notetaking, repeated practice, proper citation of sources, scientific communication, and long-term mastery of mathematics, physics, engineering, and related disciplines.

Math Monthly – Quater 2 – April/May/June – 2026

ABOUT THIS BOOK

The following was written with voice to text and rewritten with ChatGPT

This book is a collection of lessons created in order and recorded live through Zoom sessions for the PLEM Academy Research Position. Within this book, you will find a large collection of solutions covering a variety of subjects ranging from basic mathematics to advanced and higher mathematics, physics, engineering applications, and everything in between.

The purpose of this book is to provide a physical alternative to the accompanying digital website for Research Position students to review, study, and edit. Research and internship students may receive credit in future publications for finding legitimate errors, correcting typos, contributing to solutions, or completing designated research work.

For the most part, the mathematics and physics throughout this book should be sound and correct. However, this is a pre-edit research edition and may contain typos, inaccuracies, discrepancies, or incomplete information. Part of the purpose of the book is for the reader to find these problems and develop an eye for detail.

Each problem is connected to a specific textbook or source. If you find a possible error, your correction should reference the particular textbook and edition used for that lesson. You should not simply choose another textbook, website, or presentation of the concept and assume that a difference means something is incorrect. References and links to the books used are provided in the back of this book when available. Some books may have originally been found through publicly accessible online sources, and those links may change or disappear over time. Use this book to enhance your understanding of mathematics and physics, improve your problem-solving abilities, and develop greater attention to detail. You can also simply read it for enjoyment. I encourage you to treat it almost like a fiction book: sit somewhere comfortable, read through the mathematics at your own pace, explore interesting problems, and enjoy seeing how different mathematical, physical, and engineering concepts develop.

Throughout the book, you will find problems labeled [RB]. These are problems or activities for Research Position students to complete in order to build and strengthen their own skills.

You will also find problems labeled [RP]. These are Research Position problems intended for students to continue independently. You may be asked to verify an answer, expand a solution, investigate an idea, attempt a proof, or develop additional work related to the original problem. The purpose is to encourage you to do your own research and contribute your own reasoning rather than simply reading the completed work.

This book is the pre-edit version of what will eventually become a completed publication. The volumes are assembled according to the lessons, subjects, and research I am working on at the time. Research members are able to read and work with this material while it is still being developed before the final edited volume is completed.

Whether you are participating as a Research Position student, completing internship work, looking for errors and possible improvements, or simply reading because you enjoy mathematics and physics, use this book to expand your knowledge, strengthen your attention to detail, and enjoy the process of learning and research.

Enjoy my work, and keep an eye out for anything I may have missed.

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How to Use this Textbook

The objective is to teach you how to learn. More specifically, how to use textbooks properly. If you leave the textbook, you have left the subject. You might as well leave school because you are not doing math if you leave the book.

• Each lesson within this textbook is designed for you to cite another textbook and use the methods from the book to solve the question that came from the book.

• The questions are made and designed specifically for that text—that is, the properties, rules, laws, ..., definitions, theorems, ... are all linked and bounded together.

• The questions chosen are chosen to be solved under the rules of that book. If you leave the book, you have left the subject!

Instructions: You have the question and the answer from the book.

QUESTION [Book X] Chapter.Section.Problem

Question from textbook

ANSWER

Answer from textbook

The Solution is written under the double line before.

The Answer should match the book, but it may not. Sometimes the answers in the back of the book are wrong. I'd say about 1-5% of the time, the answers have typos. This is why we must use the full circle for we cannot be sure that any answer is correct unless we can verify it ourselves.

[Full Circle] This is where we begin the full circle technique(s).

Verifying the answer is correct.

GOAL Your goal should be to keep the solution covered and attempt to write your solution and see if you can get the correct answer and then verify the answer is in fact correct via the full circle of math technique(s).

After you write your solution, you then compare it to mine to see if you are on point. Making an effort to be more professional each time you attempt to solve a problem—write a solution—an English essay.

Before Starting Math

Before you begin the journey of mathematical evolution of the mind, it is important to understand what math truly is. Math is language or the application of language—linguistics.

For our PLEM Academy books, we are starting at prealgebra from a college algebra textbook—that is, the prerequisite chapter.

To begin, we will start with [19] "Sullivan Algebra & Trigonometry."

• First notice the title, "*Sullivan* Algebra & Trigonometry." Why is this important? Well, because the subject we are studying belongs to **Sullivan's Algebra** meaning that it is not the same as **Stewart's Algebra**. Meaning, the content is not universal or exchangeable without citation or reference and proof under the rule of said book. What this means is that you are studying Sullivan's Algebra—not Stewart's Algebra. These are quite literally two different subjects.

• In order to study mathematics, you must understand English (or your native language). You cannot do math if you don't understand language—English. I.e., you are writing an English essay every time you solve a problem. This means there is a title (the question), an intro (the collection of tools), a body (the application of the tools), and a summary (the answer stated with reference).

On a very basic level, we must understand what we doing when asked a question and how to address the solution.

The Full Circle of Math

Question → Solution → Answer → Proof (or showing)

I. Question (title of essay) provides the instructions.
    i. The question tells you what to do. You need to learn how to decipher the instruction.
II. Solution
    i. Introduction: The collection of formulae, equation, or anything needed to solve the problem, rather write the solution.
    ii. Body: The application of all the tools from the introduction.
    iii. Summary: Stating the answer with reference (citation).
III. Proving or Showing
    i. This is the full circle. Once you have the answer, you reverse the process one way or another to show that the answer is in face correct.

"The Full Circle of Math" should be done without technology. Once understood, then A.I. or calculators ... are fine to use! You just can't use them until you actually understand the full circle of math! NOTE: The full circle is not always necessary as somethings are just statements.

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Question Layout- Optimization

QUESTION (Title of the Essay)

ANSWER (Summary of Essay)

SOLUTION (Intro, Body, and Summary of the Essay)

OPTIMIZATION Essay Writing

$\mathcal{F}_0$ [collect data]

Keywords:

Equations Needed	Given Data	Assemble a one variable function

$\mathcal{F}_1$ [Implicitly Differentiate w/Respect to Time]

$\mathcal{F}_2$ [Summarize]

For Related Rates, we are essentially executing algorithms for physics questions in theory–meaning non-physical—precise.

We look at physics problems differently than math problems and then there is a difference between calculus application and algebraic application of derived calculus formulae found in the standard physics textbooks. What I mean by that is, in your university physics course, you will use derived algebraic equations or formulas vs. assembling functions and using calculus techniques to identify rates of change, minimums, maximums from graphical relation and definition relation. Multiple worlds of physics and math problem solving.

Optimization in Differential Calculus

A Basic Rectangle Situation

QUESTION [36] 4.7.3 (Title of the Essay)

Find two positive numbers **whose product** is 100 and **whose sum** is a minimum.

ANSWER [Page A90] (Summary of Essay)

$$10, 10$$

SOLUTION (Intro, Body, and Summary of the Essay)

OPTIMIZATION Essay Writing

$\mathcal{F}_0$ [collect data] Keywords: A rectangle- product and sum.

NOTE This question will come back in calculus 3—multivariable. You will use a 3D system to solve it.

Equations Needed	Given Data	Assemble a one variable function
$A = \text{Product} = xy$ $\text{Sum} = S = x + y$	$xy = 100$	$A = xy = 100 \quad \Rightarrow \quad y(x) = \frac{100}{x}$ $S = x + y \quad \Rightarrow \quad S(x) = x + \frac{100}{x}$

$\mathcal{F}_1$ [Implicitly Differentiate w/Respect to Time]

Now, for this question, we are asked to minimize S. Thus, we differentiate and follow max/min rules.

$$S(x) = x + \frac{100}{x} \quad \Rightarrow \quad S' = 1 - \frac{100}{x^2}$$

$$1 - \frac{100}{x^2} = 0 \quad \cancel{\Rightarrow \quad 1 = \frac{100}{x^2} \quad \Rightarrow \quad x^2 = 100}$$

[PL (Point Loss)] Do not move anything to the right side of the equation, when solving for an unknown variable. Do this- (You are looking at a function equal to zero—not an equation. S is the derivative, which is a ratio and is function, so you have to account for the denominator for vertical asymptote. Which is why you cannot(should not) execute the above red text.

$$1-\frac{100}{x^2}=\frac{x^2-100}{x^2}=0 \;\Rightarrow\; x\neq 0, \qquad x^2-100=(x-10)(x+10)=0$$

$$\Rightarrow \quad x\neq 0, \qquad x=-10 \quad \text{or} \quad x=10.$$

Since we are dealing with an optimization problem, can disregard negative values and zero. We will start with the "Second Derivative Test" for maximum and minimum values.

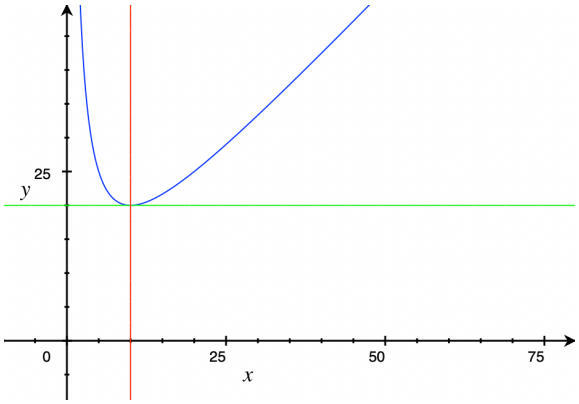
$$S''=\frac{200}{x^3} \;\Rightarrow\; s''(x=10)=\frac{200}{10^3}=\frac{10(10)(2)}{10(10)(10)}=\frac{2}{10}=\frac{1}{5}>0.$$

$$\because \; S''(10)>0, \qquad \min S(x=10)=10+\frac{100}{10}=20.$$

Then, if $x=10$. Then,

$$y(x=10)=\frac{100}{x}=\frac{100}{10}=10.$$

Are 10 and 10 the correct values? The second derivative test tells us that $x=10$ is the minimum value. So, yes. Graphically:



We can see on the graph of $y=x+100x^{-1}$, that there is a minimum at $x=10$ and the minimum value $\min y=20$.

The point of the minimum is at (10,20).

$\mathcal{F}_2$ [Summarize]

Thus, according to the second derivative (or the line test) test for maximum and minimum values, we see that the two values that minimize are 10, 10.

For Related optimization, we are essentially executing algorithms for physics questions in theory–meaning non-physical—precise.

In general, for optimization, you follow the above pattern of identifying two equations- one of multivariable nature and one of single variable nature and then put them together to make a single, single variable function for which you optimize with differential calculus techniques related to increasing/decreasing ...

A cube or box with square base

QUESTION [36] 4.7.15

If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible **volume** of the box.

ANSWER [Page A90]

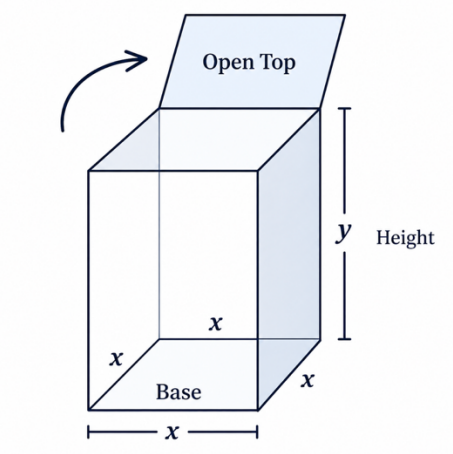
4000 cm^3

SOLUTION

$\mathcal{F}_0$ We find our two(or more) equations.

Keywords: Volume and Area

Equation 1 will be the equation related to the value given in the question. In this case, it is 1200 cm^2. Squared implies an area. So, it is referencing the surface area. There is open top, so the surface of a cuboid missing the top side.



We have for our equation equal to 1200 cm^2 as

Eq 1: $A = x^2 + 4xy = 1200$

Note- in my image, I have a lid, but the question does not specify a lid. It specifies and open top—like a terrarium. Do not include the area of the lid. Thus, only x^2.

Eq 2: $V = xxy = x^2y$

$\mathcal{F}_1$ We assemble the single variable equation in reference to what we are maximizing or minimizing. In this case, volume. Then,

$$x^2 + 4xy = 1200 \quad \Rightarrow \quad y = \frac{1200 - x^2}{4x} \quad \Rightarrow \quad V = x^2y \quad \Rightarrow \quad V(x) = x^2\left(\frac{1200 - x^2}{4x}\right)$$

$$\Rightarrow \quad V(x) = \frac{1}{4}(1200x - x^3) = 300x - \frac{1}{4}x^3.$$

$\mathcal{F}_2$ We use techniques of differentiation for max/min.

$$\frac{dV}{dx} = \frac{d}{dx}\left[300x - \frac{1}{4}x^3\right] = 300 - \frac{3}{4}x^2 = 0$$

$$\Rightarrow \quad \frac{1200 - 3x^2}{4} = 0 \quad \Rightarrow \quad 3x^2 = 1200 \quad \Rightarrow \quad x = \pm\sqrt{400} = \pm\sqrt{(4)(10)(10)} = \pm\sqrt{2^2 10^2}$$

$$= \pm\sqrt{(2 \cdot 10)^2} = \pm\, 20.$$

Since we are not concerned with negative values, we see that $x = 20$ is a possible min/max. We use the second derivative test first for speed. However, it may inconclusive.

$$\frac{d}{dx}\frac{dV}{dx} = \frac{d}{dx}\left[300 - \frac{3}{4}x^2\right] = -\frac{2}{4}(2x) = -x \quad \Rightarrow \quad V''(x = 20) = -20 < 0.$$

Thus, by the second derivative test, $V''(x = 20) < 0$ tells us there is a maximum when $x = 20$.

Now, we find the y value.
s

$$y = \frac{1200 - x^2}{4x} \quad \Rightarrow \quad y(x = 20) = \frac{1200 - (20)^2}{4(20)} = \frac{800}{80} = 10.$$

There is a maximum when $x = 20$ and $y = 10$. Now we maximum volume.

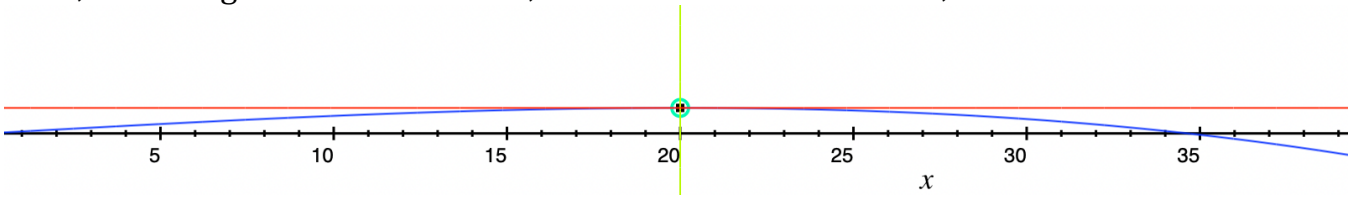
$$V(x) = x^2\left(\frac{1200 - x^2}{4x}\right) \quad \Rightarrow \quad V(x = 20) = (20)^2\left(\frac{1200 - (20)^2}{4(20)}\right)$$

Or (for speed on exams)

$$V(x, y) = x^2 y \quad \Rightarrow \quad V(x = 20, y = 10) = (20)^2(10) = 400(10) = 4000.$$

$\mathcal{F}_3$ Finalize with appropriate units.

Thus, according to differential rules, we see that the volume is 4,000 cm^3.



From our graph, we see the max is when $x = 20$ and the max value is $y = 4000$ (the volume).

Properties of Real Numbers

QUESTION [28] 1.1.13 *Title of Essay*

State the property of real numbers being used.

$$(x + 2y) + 3z = x + (2y + 3z).$$

ANSWER

Associative Property of Addition

SOLUTION

Introduction to the Essay
$\mathcal{F}_1$ [Formulae, Definition(s), and or Theorem(s)] **50% of Exam Grade** *0% without the latter*:

Look to the book for the proper terminology:

On page 3, we see from the 'Properties of Real Numbers' that the above is the associativity property.

$$(a + b) + c = a + b + c = a + (b + c).$$

Body of the Essay
$\mathcal{F}_2$ [Formulate: Assemble the Previous Data]**50% of Exam Grade** *0% without the former*:

Let $a = x,\ b = 2y,\ c = 3z.$

Then,

$$(x + 2y) + 3z = x + 2y + 3z = x + (2y + 3z).$$

Summarization and citation of Essay
$\mathcal{F}_3$ [Finalize: Clear State the Final Answer with Proper Reference] **0% of Exam Grade**:

Thus, the associative property is being used.

NOTE The previous information may seem like overkill. However, when students get to higher math, they aren't able to comprehend the questions being asked because they don't understand the terminology and how it relates to the math. It is important to absorb this information in-depth early on to engrave into one's psyche for use in higher math.

[FS (further study)] learn and show how the other properties work on page 3 or from whatever book you are using. Keep with the correct notations and definitions per your book

ODE – Dirac Delta Function and Unit Impulse Function

QUESTION [11] 7.6.7

Solve the second order ODE using method of Laplace Transform,

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0, \qquad y'(0) = 1.$$

ANSWER

$$y = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1)$$

Before we begin investigating this unique function, we must acknowledge that it is an operator and all operators have instructions; without the instructions we cannot perform said operations.

Unit Impulse Function

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases}, \qquad a > 0, t_0 > 0.$$

$$\int_0^\infty \delta_a(t-t_0) = 1.$$

The **Dirac Delta Function**: Defined by taking the limit of the **Unit Impulse** function. That is,

$$\lim_{a\to 0} \delta_a(t-t_0) = \delta_0(t-t_0) \equiv \delta(t-t_0).$$

$$\text{[i]}\ \ \delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \ne t_0 \end{cases}, \qquad \text{[ii]}\ \int_0^\infty \delta(t-t_0)\,dt = 1.$$

FOCUS DEFINITION Dirac Delta Integration

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

Sifting Property: When an object in the integrand is simply evaluated at the constant within the integrand leading to no formal arithmetic required.

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

[FE] This formula will be introduced in advance physics related (ENGR) courses junior year.

NOTE All of this is just rules and instructions based on definitions. Don't try to understand anything as it is about as productive as understanding why the letter A has the shape and sound it does—that is, someone defined it that way.

[TM] This is a function that applies in special cases and there is no need to contemplate the meaning behind it as it is simply a definition.

Laplace Transform: Dirac Delta Function

Solve the second order ODE using method of Laplace Transform,

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0, \qquad y'(0) = 1.$$

We begin by apply the Laplace transform

$$y'' + 2y' = \delta(t-1) \;\Rightarrow\; \mathcal{L}\{y'' + 2y' = \delta(t-1)\}$$

NOTE The Laplace Operator is a linear transformation.

DEFINITION **Linear Transformation – Laplace Operator is a Linear Transformation**

$$\mathcal{L}\{f(\alpha t) + g(\beta t)\} = \mathcal{L}\{f(\alpha t)\} + \mathcal{L}\{g(\beta t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$$

$$\Rightarrow\; \mathcal{L}\{y''\} + \mathcal{L}\{2y'\} = \mathcal{L}\{\delta(t-1)\} \;\Rightarrow\; \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} = \mathcal{L}\{\delta(t-1)\}$$

Now that we are at the linear transformation, we need the Laplace Transformation information.

THEREOM Laplace Transform

$$\mathcal{L}\{y\} = Y(s), \qquad \mathcal{L}\{y'\} = sY(s) - y(0), \qquad \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0), \ldots$$

or

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

$$f, f', \ldots, f^{(n-1)} \text{ are continuous on } \mathbb{R}^+ \cup \{0\} \equiv [0, \infty).$$

$$\Rightarrow\; [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] = \mathcal{L}\{\delta(t-1)\}$$

Dirac Delta Laplace Transform

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$\Rightarrow \quad [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] = e^{-st_0}$$

$$\Rightarrow \quad s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) = e^{-st_0} \text{ [solve for } Y(s)]$$

$$\Rightarrow \quad s^2Y(s) + 2sY(s) - sy(0) - y'(0) - 2y(0) = e^{-st_0}$$

$$\Rightarrow \quad s^2Y(s) + 2sY(s) = e^{-st_0} + \big(sy(0) + y'(0) + 2y(0)\big)$$

$$\Rightarrow \quad s^2Y(s) + 2sY(s) = e^{-st_0} + \big((s)(0) + (1) + 2(0)\big)$$

$$\Rightarrow \quad Y(s)(s^2 + 2s) = e^{-st_0} + 1$$

$$\Rightarrow \quad Y(s) = \frac{e^{-s(1)} + 1}{s^2 + 2s} = \frac{e^{-s} + 1}{s^2 + 2s}.$$

Now, the tricky part: We have to arrange this in a form that fits a pattern that matches a preexisting Laplace Inverse. We go to the table of Laplace Inverses (usually in the back of the textbook).

Laplace Inverse

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \qquad \mathcal{L}^{-1}\{Y(s)\} = y(t).$$

Examine the possible alterations to the existing function before making any final decisions.

$$Y(s) = \frac{e^{-s} + 1}{s^2 + 2s} = \frac{e^{-s}}{s^2 + 2s} + \frac{1}{s^2 + 2s} = e^{-s}\frac{1}{s^2 + 2s} + \frac{1}{s^2 + 2s}$$

→TYPO TO BE CORRECTED STARTING HERE← [everything above is correct]

$$= e^{-s}\frac{1}{(s+1)^2 - 2} + \frac{1}{(s+1)^2 - 2} = e^{-s}\frac{1}{(s+1)^2 - \sqrt{2}^2} + \frac{1}{(s+1)^2 - \sqrt{2}^2}$$

From the Laplace Inverse Table

$$\mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2 - k^2}\right\} = e^{at}\sinh kt$$

*BIG TYPO ↘

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{Y(s) = \frac{e^{-s}+1}{s^2+2s}\right\} \quad \Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s+1)^2-\sqrt{2}^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-\sqrt{2}^2}\right\}$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s+1)^2-\sqrt{2}^2}\right\} + \frac{1}{\sqrt{2}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s-(-1))^2-\sqrt{2}^2}\right\}$$

$$\Rightarrow \quad y(t) = f(t-1)\mathcal{U}(t-1) + \frac{1}{\sqrt{2}}\,e^{-t}\sinh\left(\sqrt{2}t\right)$$

$$= \frac{1}{\sqrt{2}}\,e^{-(t-1)}\sinh\left(\sqrt{2}(t-1)\right)\mathcal{U}(t-1) + \frac{1}{\sqrt{2}}\,e^{-t}\sinh\left(\sqrt{2}t\right).$$

[FC] To prove that the answer is in fact the correct answer, take it and then compute the derivatives and plug them into $y'' + 2y' = \delta(t-1)$ (the original equation); if the equation balances—that is, $0 = 0$, then you know you did it correctly. Hence, you do not need ChatGPT or a teacher's solution manual and you have effectively done mathematics—not puzzle solving. The time to do this, is too lengthy at this moment so we will check the answer in the back of the book but before checking the answer, make sure you take the time to acknowledge the steps of reverse engineering and proving the solution you provided that outputted the answer, is in fact the correct answer.

NOTE This was an hour and half of work and it may not be right. We check the answer in the back of the book now.

The books answer is

$$y = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1).$$

This does not mean our answer is incorrect, it may be in a different form is all. Our answer is similar but not identical. Recall the hyperbolic trigonometric function

$$\sinh(t) = \frac{e^t - e^{-t}}{2}.$$

For this question, I will leave it to you to verify whether I made a typo or if

$$\frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1) = \frac{1}{\sqrt{2}}\,e^{-(t-1)}\sinh\left(\sqrt{2}(t-1)\right)\mathcal{U}(t-1) + \frac{1}{\sqrt{2}}\,e^{-t}\sinh\left(\sqrt{2}t\right).$$

Check

$$\mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2-k^2}\right\} = e^{at}\sinh kt = e^{at}\left[\frac{e^{kt}-e^{-kt}}{2}\right] \Rightarrow \sinh\left(\sqrt{2}t\right) = \frac{e^{\sqrt{2}t}-e^{-\sqrt{2}t}}{2}$$

There answer might be wrong or mine may be wrong or they are both the same. You check on your own without ChatGPT or solution manuals. Do not give up! We will come back to this question tomorrow and go back through it to be sure and finalize the answer and verify if it is correct after you all review it as part of the researchers.

→ ALL BELOW THIS LINE IS CORRECT → Corrected version below (reference both)

Starting with

$$Y(s) = \frac{e^{-s}+1}{s^2+2s} = \frac{e^{-s}+1}{s^2+2s+1-1} = \frac{e^{-s}+1}{(s+1)^2-1} = \frac{e^{-s}+1}{\big(s-(-1)\big)^2-1}$$

From the Laplace Inverse Table

$$\mathcal{L}^{-1}\{Y\} = y, \qquad \mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2-k^2}\right\} = e^{at}\sinh kt$$

$$\mathcal{L}^{-1}\{e^{-bs}F(s)\} = f(t-b)\mathcal{U}(t-b)$$

Note that the "a" and "b" are both a in the formula book but they are different a's here.

$$Y(s) = \frac{e^{-s}+1}{s^2+2s} = \frac{e^{-s}+1}{s^2+2s+1-1}$$

$$= \frac{e^{-s}+1}{(s+1)^2-1}$$

$$= \frac{e^{-s}+1}{\big(s-(-1)\big)^2-1}$$

$$= e^{-s}\frac{1}{\big(s-(-1)\big)^2-1} + \frac{1}{\big(s-(-1)\big)^2-1}$$

$$= e^{-s}Y(s) + Y(s)$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-s}Y(s)\} + \mathcal{L}^{-1}\{Y(s)\}$$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-s}Y(s)\} + \mathcal{L}^{-1}\{Y(s)\}$$

$$= y(t-1)\mathcal{U}(t-1) + e^{-t}\sinh t$$

$$= e^{-(t-1)} \sinh(t-1)\, \mathcal{U}(t-1) + e^{-t} \sinh t$$

Hyperbolic Sine Function Operation

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad \exp\{at\} = e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} = 1 + at + \frac{a^2 t^2}{2} + \cdots$$

$$= e^{-(t-1)} \left[\frac{e^{t-1} - e^{-(t-1)}}{2}\right] \mathcal{U}(t-1) + e^{-t} \left[\frac{e^t - e^{-t}}{2}\right]$$

$$= \left[\frac{e^{(t-1)-(t-1)} - e^{-(t-1)-(t-1)}}{2}\right] \mathcal{U}(t-1) + \frac{e^{t-t} - e^{-t-t}}{2}$$

$$= \left[\frac{e^0 - e^{-2(t-1)}}{2}\right] \mathcal{U}(t-1) + \frac{e^0 - e^{-2t}}{2}$$

$$= \left[\frac{1 - e^{-2(t-1)}}{2}\right] \mathcal{U}(t-1) + \frac{1 - e^{-2t}}{2}.$$

The books answer is

$$y = \frac{1}{2} - \frac{1}{2} e^{-2t} + \left[\frac{1}{2} - \frac{1}{2} e^{-2(t-1)}\right] \mathcal{U}(t-1).$$

[FC] Now, we will complete this solution by verifying our answer is indeed correct.

Before moving forward, you should challenge your mind to see if you yourself can figure out the method to verifying if the answer is correct or not.

[FCS] For this case: all we have to do is take the answer and plug it into the original equation and see if it balances to $0 = 0$.

We must take the derivative of y twice and plug it into

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0, \qquad y'(0) = 1.$$

NOTE Just because our answer matches the solution manual, doesn't guarantee the answer is correct. The TA that solved it may very well have made the same mistake we did. You need to

know how to ensure your answers are correct without a preexisting answer and without technology before you are allowed to use technology.

FONT 11

$$y=\frac{1}{2}-\frac{1}{2}e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1)$$

$$\Rightarrow \frac{d}{dt}\left[y=\frac{1}{2}-\frac{1}{2}e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1)\right]\Rightarrow \frac{d}{dt}y=\frac{d}{dt}\frac{1}{2}-\frac{1}{2}\frac{d}{dt}e^{-2t}+\frac{d}{dt}\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1)$$

$$\Rightarrow\ y'=0-\frac{1}{2}\left(e^{-2t}\frac{d}{dt}(-2t)\right)+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\frac{d}{dt}\mathcal{U}(t-1)+\mathcal{U}(t-1)\frac{d}{dt}\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]$$

From later on, we come back to this (skip for now and come back).

$$y'=-\frac{1}{2}\left(e^{-2t}\frac{d}{dt}(-2t)\right)+\mathcal{H}'(t)$$

$$\Rightarrow\ y'=-\frac{1}{2}\left(e^{-2t}(-2)\right)+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\frac{d}{dt}\mathcal{U}(t-1)+\mathcal{U}(t-1)\left[\frac{d}{dt}\frac{1}{2}-\frac{1}{2}\frac{d}{dt}e^{-2(t-1)}\right]$$

Right now, it is not necessary to take the derivative of the $\mathcal{U}$ as it should balance in operator form just fine.

$$\Rightarrow\ y'=-\frac{1}{2}\left(e^{-2t}(-2)\right)+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1)+\mathcal{U}(t-1)\left[0-\frac{1}{2}e^{-2(t-1)}\frac{d}{dt}\left(-2(t-1)\right)\right]$$

$$\Rightarrow\ y'=e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1)+\mathcal{U}(t-1)\left[-\frac{1}{2}e^{-2(t-1)}\left(-2\frac{d}{dt}(t-1)\right)\right]$$

$$\Rightarrow\ y'=e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1)+\mathcal{U}(t-1)\left[-\frac{1}{2}e^{-2(t-1)}\left(-2\left(\frac{d}{dt}t-\frac{d}{dt}1\right)\right)\right]$$

$$\Rightarrow\ y'=e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1)+\mathcal{U}(t-1)\left[-\frac{1}{2}e^{-2(t-1)}\left(-2\left(\frac{dt}{dt}-0\right)\right)\right]$$

$$\Rightarrow\ y'=e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1)+\mathcal{U}(t-1)\left[-\frac{1}{2}e^{-2(t-1)}(-2(1))\right]$$

$$\Rightarrow\ y'=e^{-2t}+\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1)+\mathcal{U}(t-1)\left[e^{-2(t-1)}\right]$$

Now, we need y'' to balance the second-order-linear-nonhomogeneous-ODE

NOTE “homogeneous” is pronounced home-ah-gene-E-us.

[DEF] **Homogenization** (1) a process by which the fat droplets from milk are emulsified and the cream does not separate. (2) the process of making things uniform or similar.

It ain't be no poe tait toe / po taht toe because der aint no Z up inside dat "homogeneous."

$$\Rightarrow \frac{d}{dt}\left[y' = e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \mathcal{U}(t-1)\left[e^{-2(t-1)}\right]\right]$$

$$\Rightarrow \frac{d}{dt}y' = \frac{d}{dt}e^{-2t} + \frac{d}{dt}\left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) + \frac{d}{dt}\mathcal{U}(t-1)\left[e^{-2(t-1)}\right]$$

To save time, recognize that one piece in the second derivative is exactly the same as the first derivative—that is,

$$\frac{d}{dt}\left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1) = \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\frac{d}{dt}\mathcal{U}(t-1) + \mathcal{U}(t-1)\frac{d}{dt}\left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right] = \mathcal{H}'(t).$$

We don't actually need to expand ↑ because the two objects from y' and y'' will cancel regardless of the expansion. So, to save writing, let us label the above as $\mathcal{H}'(t)$.

$$\frac{d}{dt}\mathcal{U}(t-1)\left[e^{-2(t-1)}\right] = \mathcal{U}(t-1)\frac{d}{dt}\left[e^{-2(t-1)}\right] + \left[e^{-2(t-1)}\right]\frac{d}{dt}\mathcal{U}(t-1)$$

$$= \mathcal{U}(t-1)\left(-2e^{-2(t-1)}\right) + \left[e^{-2(t-1)}\right]\mathcal{U}'(t-1), \qquad \frac{d}{dt}e^{-2t} = -2e^{-2t}.$$

$$\therefore y'' = -2e^{-2t} + \mathcal{H}'(t) + \mathcal{U}(t-1)\left(-2e^{-2(t-1)}\right) + \left[e^{-2(t-1)}\right]\mathcal{U}'(t-1),$$

$$y' = e^{-2t} + \mathcal{H}'(t).$$

Assuming our derivatives are correct, we will plug and chug! For those of you watching or reading in the future, if there is a typo after we plug this in, then your job will be to find it, and put it on the website to get credit for the book in the research position certification.

$$y'' + 2y' = \delta(t-1)$$

To show something is correct, you start on the left side of the equation and move until it equals the right side of the equation.

FONT 11

$$\therefore \quad y'' + 2y' = \left[-2e^{-2t} + \mathcal{H}'(t) + \mathcal{U}(t-1)\left(-2e^{-2(t-1)}\right) + \left[e^{-2(t-1)}\right]\mathcal{U}'(t-1)\right] + 2\left[e^{-2t} + \mathcal{H}'(t)\right]$$

$$= -2e^{-2t} + \mathcal{H}'(t) + \mathcal{U}(t-1)\left(-2e^{-2(t-1)}\right) + \left[e^{-2(t-1)}\right]\mathcal{U}'(t-1) + 2e^{-2t} + 2\mathcal{H}'(t)$$

$$= -2e^{-2t} + 2e^{-2t} + \mathcal{H}'(t) - 2e^{-2(t-1)}\mathcal{U}(t-1) + e^{-2(t-1)}\mathcal{U}'(t-1) + 2\mathcal{H}'(t)$$

$$= 3\mathcal{H}'(t) - 2e^{-2(t-1)}\mathcal{U}(t-1) + e^{-2(t-1)}\mathcal{U}'(t-1)$$

$$= 3\mathcal{H}'(t) - 2e^{-2(t-1)}\left[\mathcal{U}(t-1) - \mathcal{U}'(t-1)\right]$$

$$= 3\left[\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1)+\mathcal{U}(t-1)\left[e^{-2(t-1)}\right]\right]-2e^{-2(t-1)}[\mathcal{U}(t-1)-\mathcal{U}'(t-1)]$$

$$= 3\left[\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1)+\mathcal{U}(t-1)\left[e^{-2(t-1)}\right]\right]-2e^{-2(t-1)}[\mathcal{U}(t-1)-\mathcal{U}'(t-1)]$$

$$= 3\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}'(t-1)+3\mathcal{U}(t-1)\left[e^{-2(t-1)}\right]-2e^{-2(t-1)}\mathcal{U}(t-1)+2e^{-2(t-1)}\mathcal{U}'(t-1)$$

$$= 3\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}+2e^{-2(t-1)}\right]\mathcal{U}'(t-1)+e^{-2(t-1)}\mathcal{U}(t-1)$$

$$\stackrel{?}{=}\delta(t-1)$$

Recall the **definition of the Dirac Delta Function**. We are going to reverse it.

$$\lim_{a\to 0}\delta_a(t-t_0)=\delta_0(t-t_0)\equiv\delta(t-t_0).$$

$$\text{[i]}\ \ \delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases},\qquad \text{[ii]}\ \int_0^\infty\delta(t-t_0)\,dt=1.$$

It could be right. But what is $\mathcal{U}$?

The Unit Step Function

$$\mathcal{U}(t-a)=\begin{cases}0, & 0\le t<a,\\ 1, & t\ge a.\end{cases}$$

$$= 3\left[\frac{1}{2}-\frac{1}{2}e^{-2(t-1)}+2e^{-2(t-1)}\right]\mathcal{U}'(t-1)+e^{-2(t-1)}\mathcal{U}(t-1)$$

MISTAKE ↓

The million-dollar question: What is $\mathcal{U}'=?$

NOTE Had we acknowledged the "Unit Step Function" first. We would have saved ourselves a lot of tedious calculation.

$$y'=e^{-2t}+\frac{1}{2}\frac{d}{dt}\mathcal{U}(t-1)-\frac{1}{2}\frac{d}{dt}e^{-2(t-1)}\mathcal{U}(t-1)$$

$$= e^{-2t}+\frac{1}{2}\frac{d}{dt}\mathcal{U}(t-1)-\frac{1}{2}\left(e^{-2(t-1)}\frac{d}{dt}\mathcal{U}(t-1)+\mathcal{U}(t-1)\frac{d}{dt}e^{-2(t-1)}\right).$$

Since the Unit Step Function is a piecewise-constant function, where its derivative is 0 everywhere except when $t = t_0$. Hence,

$$y' = e^{-2t} + e^{-2(t-1)}\mathcal{U}(t-1)$$

That looks a heck of a lot easier to work with!

$$y'' = \frac{d}{dt}e^{-2t} + \frac{d}{dt}e^{-2(t-1)}\mathcal{U}(t-1)$$

$$\Rightarrow \quad y'' = -2e^{-2t} + e^{-2(t-1)}\frac{d}{dt}\mathcal{U}(t-1) + \mathcal{U}(t-1)\frac{d}{dt}e^{-2(t-1)}$$

$$\Rightarrow \quad y'' = -2e^{-2t} - 2e^{-2(t-1)}\mathcal{U}(t-1)$$

Check answer

$$y'' + 2y' = \left[-2e^{-2t} - 2e^{-2(t-1)}\mathcal{U}(t-1)\right] + 2\left[e^{-2t} + e^{-2(t-1)}\mathcal{U}(t-1)\right]$$

$$= -2e^{-2t} - 2e^{-2(t-1)}\mathcal{U}(t-1) + 2e^{-2t} + 2e^{-2(t-1)}\mathcal{U}(t-1)$$

$$= 0.$$

So, does

$$0 = \delta(t-1)?$$

Recall part (i) of the definition of the Dirac Delta function.

$$\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}.$$

There are two t_0. From the initial conditions, we have $y(t_0) = y_0,\ y'(t_0) = y_0'$ where $t_0 = 0$.

We then have $\delta(t - t_0)$ where $t_0 = 1$. These are two different conditions not to be mixed up just like the a, b in the exponent earlier with the transform formulae.

INCORRECT BELOW ↓

It would appear, everywhere $t \neq 1$, the solution is correct. Since this is an initial condition problem, where we have $y(0) = 0, y'(0) = 1$. This tells us $t_0 = 0$. And since $t_0 = 1 \neq 0$. $\delta(t - t_0) = 0$. Banger!

Yes, the answer is $0 = 0$. It balances. This shows that our solution was correct because the answer balanced the equation. That still does not mean it is 100% guaranteed. We would require an eye from team of people to fully concur that it is correct. Even still, it may not be

correct. A famous paper that knighted Andrew Wiles for proving Fermat's Last Theorem, turned out to have a big typo and it was reviewed by many peers. It was still correct but wasn't.

INCORRECT ABOVE ↑

In the above, I has stated that the t_0 is the initial condition from the IVP but the actual t_0 is coming from the Dirac Delta condition $\delta(t-1)$ where $t_0 = 1$.

Now, from $0 = \delta(t-1)$, we know that $t \in [0,\infty),\ t_0 > 0$ from the initial definitions.

So, for the Dirac Delta,

$$\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases},$$

We can say with confidence $0 = \delta(t-1)$ is true iff $t \neq 1$. Thus, the correct answer for this question should be

$$y(t) = e^{-(t-1)} \sinh(t-1)\, \mathcal{U}(t-1) + e^{-t} \sinh t\,, \qquad t \in [0,1) \cup (1,\infty).$$

Is this 100% correct statement?

Equations and Inequalities

Quadratic Formula – Complex Solution

QUESTION [19] 1.CR.77

Solve each equation in the complex number system.

$$x(1-x) = 6$$

ANSWER

$$\left\{\frac{1}{2} - \frac{\sqrt{23}}{2}i, \frac{1}{2} + \frac{\sqrt{23}}{2}i\right\}$$

DEFINITION Imaginary Unit, i

$$i^2 = -1.$$

DEFINITION Complex Numbers

$$z = a + bi = \mathrm{Re}\{z\} + \mathrm{Im}\{z\}\sqrt{-1}, \qquad a, b \in \mathbb{R}, \qquad i = \sqrt{-1}.$$

NOTE a is the real part and b is the imaginary part and i is the imaginary unit of the complex number

$$\therefore x(1-x)=6 \;\Rightarrow\; x-x^2=6 \;\Rightarrow\; x-x^2-6=6-6 \;\Rightarrow\; x-x^2-6=0$$

$$\Rightarrow\; -x^2+x-6=0 \;\Rightarrow\; (-1)[-x^2+x-6=0] \;\Rightarrow\; x^2-x+6=0.$$

Since we are told it is complex, probably best to use quadratic formula.

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(6)}}{2(1)}=\frac{1\pm\sqrt{-23}}{2}=\frac{1}{2}\pm\frac{i\sqrt{23}}{2}.$$

For max credit: we must state the answer based on the book's notation/request. In this book, they want it as a 'solution set.' I.e.,

$$\left\{\frac{1}{2}-\frac{\sqrt{23}}{2}i,\frac{1}{2}+\frac{\sqrt{23}}{2}i\right\}.$$

Common Symbols and Notations

Before you begin algebra in college or if you have already taken it, make sure you are familiar with basic notations, rules, properties, axioms, definitions, ..., and theorems. The prerequisite section of a precalculus or algebra textbook is probably the most overlooked yet most important part of any math book. It is like the foundation of a house. We will begin with some basic notations to clear up confusion.

Number Sets

Complex Number	Real Number	Rational Number	Integer Number
$\mathbb{C}=\mathcal{C}=C=\mathrm{C}=\cdots$	$\mathbb{R}=\mathcal{R}=R$	$\mathbb{Q}=\mathcal{Q}=Q$	$\mathbb{Z}=\mathcal{Z}=Z$
$\{a+bi\mid a,b\in\mathbb{R}\}$	$x\in(-\infty,\infty)$	$\left\{\frac{a}{b}\middle\vert b\neq 0,a,b\in\mathbb{R}\right\}$	$\{\dots,-2,-1,0,1,2,\dots\}$

Irrational numbers such as π and e or $\sqrt{2}$ are written as all rational numbers removed from the real number set. I.e., $\mathbb{I}=\mathbb{R}\setminus\mathbb{Q}$ (sometimes I is the set of integers). That is, the rational numbers are removed from the real number set, leaving only irrational numbers.

The Natural Numbers, $\mathbb{N}=\{1,2,3,\dots\}$ and the positive integer set is, $\mathbb{Z}^+=\{1,2,3,\dots\}$.

The natural numbers or positive integers with zero would be $\mathbb{Z}^+\cup\{0\}=\{1,2,3,\dots\}\cup\{0\}$.

Common Symbols and Notations

Implication symbol, \Rightarrow	$\Rightarrow$	$a+b=0\;\Rightarrow\;b=-a$
In (element of), \in	$\in$	$1\in\{1,2\}$
Proper subset (contained in), \subseteq	$\subseteq$	$\{1\}\subseteq\{1,2\}$

Subset, \subset	$\subset$	$\{1,2\} \subset \{1,2\}$
For all, \forall	$\forall$	$\forall_x \in [1,2]$
If and only if (iff), \Leftrightarrow	$\Leftrightarrow$	$p \Rightarrow q \Leftrightarrow q \Rightarrow p$
Such that, \ni	$\ni$	
There exists, \exists	$\exists$	
Therefore, \therefore	$\therefore$	
Because, \because	$\because$	
Equivalent, \equiv	$\equiv$	$\mathbf{x} \equiv \vec{x}$
Goes to, \to	$\to$	$x \to a$ as $f(a) \to L$
Union, \cup	$\cup$	$\{a\} \cup \{b\} = \{a, b\}$
Intersection, \cap	$\cap$	$\{a, b\} \cap \{b, c\} = \{b\}$

The above symbols are quantifiers of sorts to condense English. There are very formal ways of using them and informal ways. Mathematicians are trained during their junior/senior years to use them formally; in the same way an English professor learns the structure of language. Physicists and engineers usually use them informally—slang. There is a place for formality and a place for slang use. You will learn as you progress.

How to Approach Limits in Calculus

QUESTION [7] 2.PE.23

Find the limit or explain why it does not exist,

$$\lim_{x\to 0} \frac{8x}{3\sin x - x}.$$

ANSWER

4

First plug $x = 0$ in

$$\lim_{x\to 0} \frac{8x}{3\sin x - x} = \frac{8(0)}{3\sin(0) - (0)} = \frac{0}{0} = \text{indeterminant.}$$

Since the limit is in indeterminate form, $\frac{0}{0}$, we could use L'Hospital's Rule if allowed. But we are not allowed yet to do this. So, we must manipulate the function.

The question is: What are our options? We begin by building an algorithmic list:

[1] Is the limit in indeterminant form?
[2] Can I factor?

$$\lim_{x\to 0}\frac{8x}{3\sin x - x} = \lim_{x\to 0}\frac{x(8)}{x\left(\frac{3\sin x}{x} - 1\right)} = \lim_{x\to 0}\frac{8}{\frac{3\sin x}{x} - 1}$$

Factoring is the correct approach. Now, we must use limit laws for ratio and then sum and difference and then look at a preexisting formula.

$$= \frac{\lim_{x\to 0} 8}{\lim_{x\to 0}\left(\frac{3\sin x}{x} - 1\right)} = \frac{\lim_{x\to 0} 8}{\lim_{x\to 0}\left(\frac{3\sin x}{x}\right) - \lim_{x\to 0}(1)} = \frac{8}{\lim_{x\to 0}\left(\frac{3\sin x}{x}\right) - 1}$$

Identify

Theorem 7 from Thomas' Calculus

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

[RP] Prove (or show) Theorem 7 is correct.

Thus,

$$= \frac{8}{\lim_{x\to 0}\left(\frac{3\sin x}{x}\right) - 1} = \frac{8}{3\lim_{x\to 0}\left(\frac{\sin x}{x}\right) - 1} = \frac{8}{3-1} = \frac{8}{2} = 4.$$

EXAM (for max credit) Find the limit or explain why it does not exist,

$$\lim_{x\to 0}\frac{8x}{3\sin x - x}.$$

Student Exam Solution

First

$$\lim_{x\to 0}\frac{8x}{3\sin x - x} = \frac{8(0)}{3\sin(0) - (0)} = \frac{0}{0} = \text{indeterminant.}$$

Second (identify approach)

$$\lim_{x\to 0}\frac{8x}{3\sin x - x} = \lim_{x\to 0}\frac{x(8)}{x\left(\frac{3\sin x}{x} - 1\right)} = \lim_{x\to 0}\frac{8}{\frac{3\sin x}{x} - 1}$$

Citation for this solution (English Essay) By **Theorem 6**, (worth 25-50% of the grade)

$$\text{T6: } \lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Thus,

$$= \frac{8}{\lim_{x\to 0}\left(\frac{3\sin x}{x}\right) - 1} = \frac{8}{3\lim_{x\to 0}\left(\frac{\sin x}{x}\right) - 1} = \frac{8}{3(1) - 1} = \frac{8}{2} = 4.$$

Student Solution

$$\lim_{x\to 0}\frac{8x}{3\sin x - x} = \frac{8(0)}{3\sin(0) - (0)} = \frac{0}{0} = \text{indeterminant.}$$

Second (identify approach)

$$\lim_{x\to 0}\frac{8x}{3\sin x - x} = \lim_{x\to 0}\frac{x(8)}{x\left(\frac{3\sin x}{x} - 1\right)} = \lim_{x\to 0}\frac{8}{\frac{3\sin x}{x} - 1}, \qquad \text{T7: } \lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Thus,

$$= \frac{8}{\lim_{x\to 0}\left(\frac{3\sin x}{x}\right) - 1} = \frac{8}{3\lim_{x\to 0}\left(\frac{\sin x}{x}\right) - 1} = \frac{8}{3(1) - 1} = \frac{8}{2} = 4.$$

NOTE Linear Algebra is the study of vector space; matrices are just tools or notations to simplify calculations more or less....

We will start by organizing notation.

Trig/Physics/3D CALC	Linear Algebra	Higher Math (in general)
$\vec{v} = \langle x, y, z\rangle$	$\mathbf{x} = (x_1, x_2, x_3, \dots)$	$x \equiv \vec{x} \equiv \mathbf{x} = (x_1, x_2, x_3, \dots)$

The above vector notations are all the same. In Quantum mechanics, the column vector is $v = |x, y, z\rangle$. All vectors can be considered column vectors in the context of linear algebra vector and matrix notation. Now, let's use the notation to assemble the matrix notation.

Solve $ax + by = c$ and $dx + ey = f$ by putting them into standard matrix equations.

First, I will index everything. $a_{11}x_1 + a_{12}x_2 = b_1$ and $a_{21}x_1 + a_{22}x_2 = b_2$.

Recall the "dot product" and(or) "inner product," $(a, b)\cdot(c, d) = ac + bd = (a, b)^T(c, d)$.

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{matrix} \Rightarrow \begin{matrix} (a_{11} \quad a_{12})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = b_1 \\ (a_{21} \quad a_{22})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = b_2 \end{matrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Coefficient Matrix	Variable Vector	Constant Vector
$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$	$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

The matrix equation, $A\mathbf{x}=\mathbf{b}$	Augmented Matrix, $[A\|\mathbf{b}]$
$A\mathbf{x}=\mathbf{b} \Leftrightarrow \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$	$[A\|\mathbf{b}] \Leftrightarrow \left[\begin{array}{cc\|c}a_{11} & a_{12} & b_1\\ a_{21} & a_{22} & b_2\end{array}\right]$

The span of set of standard column vectors is the **linear Combination** which ultimate equals $A\mathbf{x}$.

Generalized Definition "Span of Set" (changes book to book)

$$\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\dots\}=x_1\mathbf{v}_1+x_2\mathbf{v}_2+\cdots=\mathbf{v}_1x_1+\mathbf{v}_2x_2+\cdots=[\mathbf{v}_1\mid\mathbf{v}_2\mid\cdots]\begin{bmatrix}x_1\\ x_2\\ \vdots\end{bmatrix}=A\mathbf{x}.$$

From the above, we would see that $\mathbf{v}_1=\begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix},\mathbf{v}_2=\begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}$. The column vectors of A.
Show that the span of $\mathcal{V}=\{\mathbf{v}_1,\mathbf{v}_2\}$ is $A\mathbf{x}$.

$$\text{span}\mathcal{V}=\text{span}\{\mathbf{v}_1,\mathbf{v}_2\}=x_1\mathbf{v}_1+x_2\mathbf{v}_2=\mathbf{v}_1x_1+\mathbf{v}_2x_2=\begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}x_1+\begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}x_2$$

$$=\left[\begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}\mid\begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}\right]\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\mathbf{x}=A\mathbf{x}.$$

Other forms:

$$\begin{pmatrix}a_{11}x_1\\ a_{21}x_1\end{pmatrix}+\begin{pmatrix}a_{12}x_2\\ a_{22}x_2\end{pmatrix}=\begin{pmatrix}a_{11}x_1+a_{12}x_2\\ a_{21}x_1+a_{22}x_2\end{pmatrix}=\begin{bmatrix}\begin{pmatrix}a_{11}\\ a_{12}\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\end{pmatrix}\\ \begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\end{pmatrix}\end{bmatrix}=\cdots$$

Linear Algebra is most definitely the art of giving the same thing different names. You will be asked to find the same solution over and over again, but it will be worded differently, and you will want the final answer stated a certain way, even though all the math/arithmetic is the same. E.g., finding the kernel, nullspace, solution to $A\mathbf{x}=\mathbf{0}$, the orthogonal matrix, ... are all the same solution methods but are written differently, such as a vector solution, set solution, general solution, parametric solution, ... All the same arithmetic and numbers but written differently. **In this subject, all vectors are considered column vectors unless they are specified to be row vectors or are transposed**. Transposing a matrix (vector) turns a row/column into a column/row.

"Mathematics is the art of giving the same name to different things."—Henri Poincaré

Introduction to Polynomial Vector Notation in Linear Algebra

$$\mathcal{P}=\left\{a_0+a_1t+a_2t^2+\cdots+a_nt^n\middle|a\in\mathbb{R},n\in\mathbb{Z}^+\cup\{0\}\right\}$$

Let's look at a basic quadratic equation in terms of vector notation.

$$f(x) = x^2 - x + 1 = (x^2)(1) + (x^1)(-1) + (x^0)(1) = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}.$$

$$f(x) = x^2 - x + 1 = (1)(x^2) + (-1)(x^1) + (1)(x^0) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Since we are usually seeking column vectors, $\begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, would be the better choice.

We can now rewrite the vector with respect to the basis, $\mathbb{P}^2$. The following notations may vary book to book or course to course. Make sure to check your textbook and ask your professor which notation is preferred. I will use $\mathcal{P}_n$ with the index implying the number of objects in the set vs. the exponent of the leading term.

$\mathbb{P}^n \equiv \mathbb{P}_n$	$\mathcal{P}^n \equiv \mathcal{P}_n$	$\mathbb{P}^n \equiv \mathcal{P}_{n+1}$
$\mathbb{P}^2 = \{1, x, x^2\} = \mathbb{P}_2$	$\mathcal{P}^2 = \{1, x\} = \mathcal{P}_2$	$\mathbb{P}^2 = \{1, x, x^2\} = \mathcal{P}_3$

$$\therefore f(x) = x^2 - x + 1 = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{P}_3} = \mathbf{f}.$$

We now have the vector notation, $\mathbf{f} = (1, -1, 1)_{\mathcal{P}_3}$. Now, let's look at this in terms of matrices. Determine if the set, $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \{1 + t, t - 1, t^2\}$ is linearly dependent. Do the math two different ways: (1) by spanning the set and seeing if the trivial solution is the only solution, and (2) by inspection.

$$\text{span}\{1 + t, t - 1, t^2\} = (1 + t)x_1 + (t - 1)x_2 + (t^2)x_3$$

$$= \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_3$$

$$= \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_3 \right] = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathcal{P}_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{\sim}{R_1 + (-1)R_2 \Leftarrow R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathcal{P}_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We can now see that the vectors are linearly independent, since the matrix is full rank.

Section 1.3/1.4 Standards and Units / Using and Converting Units

QUESTION [3] 1.3 How many **nanoseconds** does it take light to **travel** 1.00 ft in a vacuum? (This result is a useful quantity to remember.)

ANSWER 1.02 ns

• **Keywords & sig-figs**, then [**nanoseconds=time**] [**travel=distance**] [1.00 ft = 3 sig figs]

[1st – Extract Data] 50% of the credit on exams.

Given (directly give)	**Need** (indirectly give)	**Seeking**
$\Delta x = 1.00 \text{ ft}$	(rate)(time) = (distance)	$\Delta t = ?$
$c = v_x = 299792458 \frac{\text{m}}{\text{s}}$	$v_x \Delta t = \Delta x \Rightarrow \Delta t = \frac{\Delta x}{v_x}$	

[2nd – Formulate Data] 50% of the credit on exams. Before doing any calculation, it is good to convert the units without approximating. If I just plug in what I have, the units do not match up. I will do this all-in-one shot. Without looking up the unit conversions, I am going to go off of what I remember—that is, 2.54 cm = 1 in, 100 cm = 1 m, and 12 in = 1 ft. (I had to look up in the book nanosecond conversion.)

$$\Delta t = \frac{\Delta x}{v_x} = \frac{1.00 \text{ ft}}{299792458 \frac{\text{m}}{\text{s}}} = \frac{\frac{(1.00 \text{ ft})}{1}\left(\frac{12 \text{ in}}{1 \text{ ft}}\right)\left(\frac{2.54 \text{ cm}}{1 \text{ in}}\right)\left(\frac{1 \text{ m}}{100 \text{ cm}}\right)}{299792458 \frac{\text{m}}{\text{s}}}$$

$$= \frac{\frac{(12)(2.54)}{100} \text{ m}}{299792458 \frac{\text{m}}{\text{s}}} = \frac{(12)(2.54)}{[(100)(299792458)]} \text{ s}, \qquad 1 \text{ ns} = 10^{-9} \text{ s} \Rightarrow 1 = \frac{1 \text{ ns}}{10^{-9} \text{ s}}$$

$$= \frac{(12)(2.54)}{[(100)(299792458)]} (1) \text{ s} = \frac{(12)(2.54)}{[(100)(299792458)]} \text{ s}\left(\frac{1 \text{ ns}}{10^{-9} \text{ s}}\right)$$

$$= \frac{(12)(2.54)}{[(100)(299792458)(10^{-9})]} \text{ ns} \approx 1.0167.$$

[3rd – State the Answer] 0% of the credit on exams (but no credit without it).

$$\therefore \Delta t = 1.02 \text{ ns}.$$

Quick Intro to Transpose, Rank, and Linear Dependence, Consistency

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T = [a \quad b \quad c], \qquad [a \quad b \quad c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

Also, familiarize yourself with unique, infinite, and no solution, and consistent/inconsistent:

Unique (Consistent)	Infinite Solutions (Consistent)	No Solution (Inconsistent)
$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 1 & * \end{array}\right]$ Intersection of two lines	$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 0 & 0 \end{array}\right]$ Line Multiples	$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 0 & 1 \end{array}\right]$ Parallel Lines
$\left[\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$ Intersection of 3 planes	$\left[\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$ 1 or More Plane Multiples	$\left[\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right]$ Parallel Plane(s)

The matrices above are in REF (Row Echelon Form) and show the solution based on the pivots. Next, we will understand pivots, matrix size and dimensions, what the vectors span, and so on.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Full}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Full Row Rank}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ Full Column Rank}$$

The rank is equal to the number of pivots or linearly independent columns or the dimension, ... The size of a matrix is the number of rows and columns—that is, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is 2×3 matrix.

$$A_{2\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

This matrix has rank at most equal to the number of rows.

$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(B) = 2 = \dim B$. This also means the column vectors span $\mathbb{R}^2$.

There are two rows and three columns. A set of column vectors is said to be linearly independent or linearly dependent. There are many definitions for this.

Linearly Dependent Set	Linearly Independent Set
Possible Column Vectors of B $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \{\mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_3\}$ There are many ways to see this relation...	Row Vectors of B $\{\mathbf{w}_1, \mathbf{w}_2\} = \left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T\right\}$

At this point in Linear Algebra, make sure you drill in your head that every time you are asked a question, it will (in general) lead to the same solution—that is, the answer will just be stated differently, but the method arriving at the answer is the usually same matrix reduction RREF method. I.e., if you are in doubt on an exam, simply put everything in a matrix and solve.

[PL] **Students who do not thoroughly read the Linear Algebra textbook do terribly on** exams because they don't know what the words mean when a true/false section appears. Linear Algebra is more of an English course than a math course.

It should also be noted that whether you are working with vectors, matrices, polynomials, a set of differential equation solutions, or any other set that spans a linear combination is the same concept of arithmetic. E.g., is $\{e^x, e^{-x}, -e^x\}$ a linearly independent set? There are different ways to look at this and define it, but the most common way would be to say that the trivial solution is the only solution to $A\mathbf{x} = \mathbf{0}$—trivial meaning $\mathbf{x} = \mathbf{0}$. **Linearly Dependent**: one or more objects are a multiple or sum of the other objects. **Linearly Independent**: no object is a multiple or sum of the other objects in the set.

NOTE The definitions in the section (outside of the referenced textbooks) are generalized notations and descriptions that are usually universal but may vary course to course.

Show that $\{e^x, e^{-x}, -e^x\}$ is a linearly dependent set.

$$\text{span}\{e^x, e^{-x}, -e^x\} = c_1e^x + c_2e^{-x} + c_3(-e^x) = c_1e^x + c_2e^{-x} + c_4e^x$$

$$= (c_1 + c_4)e^x + c_2e^{-x} = c_5e^x + c_2e^{-x} \quad \Rightarrow \quad \{e^x, e^{-x}, -e^x\} \sim \{e^x, e^{-x}\}.$$

Thus, the set $\{e^x, e^{-x}, -e^x\}$ is a linearly dependent set.

Introduction to Problem Solving Techniques: College Algebra

Before we begin the practice of solving a question, in the live feed with the research students, we also reviewed the textbook thoroughly. After a review of the notations and location of various tools, we chose the following question(s).

NOTE This is the initial solving for Algebra—specifically: to understand the importance of keeping one's notations to the assigned book. If you find yourself in a situation with a textbook, you must backtrack the sources/citations, your professor used or uses to draw their work from. Then, you will the book(s) you are supposed to use.

Using book [2] (found in the back of the book is the list of referenced textbooks),

"[2] Barnett, R., Ziegler, M., Byleen, K., & Sobecki, D. (2015). College Algebra with Trigonometry (9th ed.). McGraw-Hill Education. This book was found as a PDF via:

https://marinazahara22.wordpress.com/wp-content/uploads/2013/10/college_algebra_with_trigonometry.pdf"

Asymptotes in Algebra

QUESTION 4.4.75 [2]

Find all vertical, horizontal, and oblique asymptotes.

$$f(x) = \frac{x^3}{x^2+1}.$$

ANSWER

The book skipped or did not add the answer for 4.4.75. We will still answer the question together.

SOLUTION

NOTE We are not in calculus so the techniques in calculus are not to be used, and you must think the same when you get to calculus that you may not be able to use the algebra. You can find the asymptotes in calculus by using limits and evaluate infinite limits. E.g.,

$$\lim_{x\to\infty}\frac{x^3}{x^2+1} = \frac{\infty}{\infty} = \text{indeterminate} = \lim_{x\to\infty}\frac{(1)}{\left(\frac{1}{x}+\frac{1}{x^3}\right)}\cdot\frac{x^3}{x^3} = \lim_{x\to\infty}\frac{(1)}{\left(\frac{1}{x}+\frac{1}{x^2}\right)}\cdot(1)$$

$$= \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{1}{x^2}} = \frac{1}{\frac{1}{\infty}+\frac{1}{\infty}} = \frac{1}{0+0} = \frac{1}{0} = \text{undefined} \;\Rightarrow\; \lim_{x\to\infty}\frac{x}{\left(1+\frac{1}{x^2}\right)}\cdot\frac{x^2}{x^2} = \frac{\infty}{1+\frac{1}{\infty^2}}$$

$$= \frac{\infty}{1+0} = \infty. \quad \therefore \lim_{x\to\infty}\frac{x^3}{x^2+1} = \infty \;\Rightarrow\; \text{no horizontal asymptotes.}$$

Or,

$$\lim_{x\to\infty}\frac{ax^3}{bx^3+1} = \frac{a}{b} \;\Rightarrow\; \text{HA: } y = \frac{a}{b}, \qquad \lim_{x\to\infty}\frac{ax^2}{bx^3+1} = 0 \;\Rightarrow\; \text{HA: } y = 0.$$

Now, back to the algebra techniques:

What is a **rational function**? According to Def. 1 from [2]—verbatim, "A rational function is the ratio of two polynomials." Then, do you know what a polynomial is—by definition.

Now, for Algebra, we look to formula in the book.

$\mathcal{F}_1$ [Formulae]

THEOREM 2 [rational function]

If a is a zero of $q(x)$, then the line $x = a$ is a vertical asymptote of the graph of f.

$$f(x) = \frac{p(x)}{q(x)}$$

Meaning: there is a value for x for which the denominator is zero.

---

[i] For vertical asymptote, our provided function has no values in $\mathcal{R} \equiv \mathbb{R}$ (real numbers), so there are no vertical asymptotes.

[ii] For horizontal asymptote, we have the case from "Theorem 3 [Page 304]," that $m = 3 > 2 = n$. Thus, there are no horizontal asymptotes.

[iii] For oblique, "Theorem 4 [page 308]," we see that we can get the function into the form

$$f(x) = \frac{p(x)}{q(x)} = mx + b + f(x) = \frac{r(x)}{q(x)} \quad \Rightarrow \quad \text{OA: } y = mx + b \quad \Leftrightarrow \quad \frac{r^m}{q^n}, m < n.$$

To get it into this form, use long division or synthetic division. I am going to use Wolfram Alpha to perform the decomposition of the function. I can also partial fractions as well. Your choice. It is basic arithmetic; I often have Wolfram Alpha's algorithm do basic arithmetic.

$$y = \frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1} = (1)x + (0) - \frac{x}{x^2 + 1}, \qquad m = 1 < 2 = n$$

NOTE(Future) In calculus, you can't just use the horizontal asymptote formula. You have to show the steps as in the previous notes.

$\mathcal{F}_2$ [Formulate]

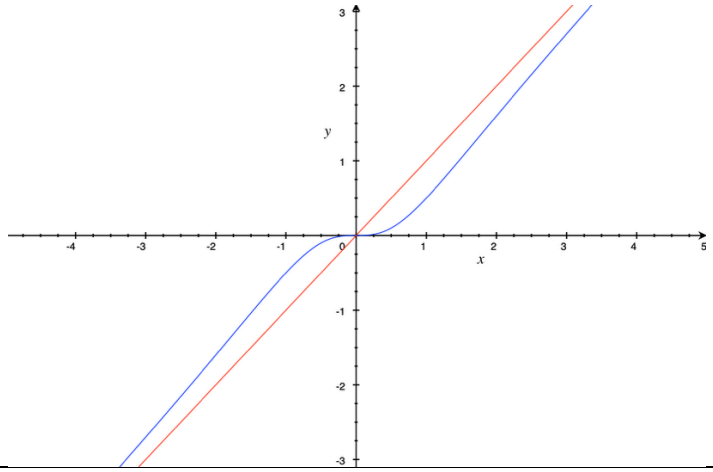
$$\frac{p(x)}{q(x)} = mx + b + f(x) = \frac{r(x)}{q(x)} \quad \Rightarrow \quad y = x$$

Graph of

$$f(x) = \frac{x^3}{x^2+1}, \qquad y = x.$$

Is this an oblique asymptote?

Using a graph is not a solution or answer. You must reference the theorem(s).



$\mathcal{F}_3$ [Finalize]

To finalize, we can see clearly that we got the correct answer (maybe)...does it satisfy the theorem? Because the answer to this question is void in the back and it is an odd question. That being said, there may be a mistake that is common that people don't catch.
The Theorem says, $[f(x) - (mx + b)] \to 0$ as $x \to \pm\infty$.

Thus, [if commit the following work because it seems redundant, you will miss half the credit]

$$[f(x) - (mx + b)] = \left[\frac{x^3}{x^2+1} - [(1)x + 0]\right] = \left[\frac{x^3}{x^2+1} - x\right]$$

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm cb}{bd}, \qquad \frac{x^3}{x^2+1} - \frac{x}{1} = \frac{(x^3)(1) - (x)(x^2+1)}{(x^2+1)(1)} = \frac{x^3 - [x^3 + x]}{x^2+1}$$

$$= \frac{x^3 - x^3 - x}{x^2+1} = \frac{-x}{x^2+1} \to 0 \quad \text{as} \quad x \to \pm\infty \quad \because n = 2 < 1 = m.$$

Thus, by definition and theorem, we see that $f(x)$ has an oblique asymptote when $y = x$.

COLLEGE EXAM METHOD *The following should be done with paper/pencil for practice exams.*

VA: $q(x) = 1 + x^2 = 0 \Rightarrow x = \pm i$. Thus, there are no real solutions. Therefore, by theorem, there are no vertical asymptotes.

HA: For horizontal asymptote, we have the case from "Theorem 3 [Page 304]," that $m = 3 > 2 = n$. Thus, there are no horizontal asymptotes.

OA: The Theorem says, $[f(x) - (mx + b)] \to 0$ as $x \to \pm\infty$.

Thus, [if commit the following work because it seems redundant, you will miss half the credit]

$$[f(x)-(mx+b)]=\left[\frac{x^3}{x^2+1}-[(1)x+0]\right]=\left[\frac{x^3}{x^2+1}-x\right]$$

$$\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm cb}{bd},\qquad \frac{x^3}{x^2+1}-\frac{x}{1}=\frac{(x^3)(1)-(x)(x^2+1)}{(x^2+1)(1)}=\frac{x^3-[x^3+x]}{x^2+1}$$

$$=\frac{x^3-x^3-x}{x^2+1}=\frac{-x}{x^2+1}\to 0\quad\text{as}\quad x\to\pm\infty\quad\because n=2<1=m.$$

Thus, by definition and theorem, we see that $f(x)$ has an oblique asymptote when $y=x$.

QUESTION Are there ways to approach proving(showing) our answer(s) is/are indeed correct under the full circle of math?

Introduction to Problem Solving Techniques: Trigonometry

Before we start trigonometry, we should understand that it is simply a continuation of algebraic operation. It is imperative that you abandon the calculator around college algebra and higher. It is okay for approximation(sometimes) in physics but usually not necessary.

The reason for the abandoning of the calculator is to understand the construct of the language. For example: What is e^x ? There are many forms but the most useful(common) is the series:

$$e^x\equiv\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots$$

This then leads me to trigonometric functions such as the cosine and sine functions, which also have many forms.

$$\sin x=?$$

Because in higher math, you are not going to simply plug things into calculators—that is, it should be considered early on what operator functions actually mean. For example, the hyperbolic sine function,

$$\sinh x=\frac{e^x-e^{-x}}{2}=\frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}-\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\right).$$

For this introduction section, we will be using the following textbook for a review lesson found via AuthorJonD.com for the research team. There is an open-source recap lesson for non-research members as well.

[19] Sullivan, M. (2011). Algebra and Trigonometry (9th ed.). Pearson.

https://home.ufam.edu.br/andersonlfc/Nivelamento_Matem%C3%A1tica/Algebra%20&%20Trigonometry%20-%20Sullivan/Sullivan%20Algebra%20&%20Trigonometry%209th%20txtbk.pdf

We will begin with a very basic question in relation to notation and calculation with a calculator. The objective in this book is to begin to see the importance of in-depth understanding of foundational notation.

QUESTION [19] 8.2.23 Find the exact value of the expression,

$$\sin^{-1}\left[\cos\left(-\frac{7\pi}{6}\right)\right].$$

ANSWER [Page AN52]

$$-\frac{\pi}{3}$$

SOLUTION

NOTE [PL] The arcsine or sine inverse notation is often confused with a reciprocal—that is,

$$f^{-1}(x) = \sin^{-1} x \neq [\sin x]^{-1} = \frac{1}{\sin x}, \qquad f^{-1}(x) = \ln^{-1} x \neq [\ln x]^{-1} = \frac{1}{\ln x},$$

$$\arcsin x \equiv \sin^{-1} x\,, \qquad \sin^n(x) = [\sin x]^n \;\Leftrightarrow\; n \neq -1.$$

Also,

$$f^{-1}\big(f(x)\big) = x = f\big(f^{-1}(x)\big).$$

$\mathcal{F}_1$ [Formulae] Let,

$$\Phi = \sin^{-1}\left[\cos\left(-\frac{7\pi}{6}\right)\right] = \arcsin\left[\cos\left(-\frac{7\pi}{6}\right)\right] = \sin^{-1}\theta$$

We may not need a definition or theorem for this question—that is, arithmetic and geometric relation with logic and reasoning may be enough; however, we may need to reference a domain/range.

NOTE The reason why I chose this question is because it comes back in a huge way in integral calculus and higher. You WILL NOT be able to use calculators.

We will start systemically with $\cos\left(-\frac{7\pi}{6}\right)$. Now, the question is how do we choose to evaluate? Do we use, "The Unit Circle," or do we use a different technique?

$\mathcal{F}_2$ [Formulate]

Identify $x = r\cos\theta$. Cosine is an even function, meaning $f(-x) = f(x)$. Thus,

$$\frac{x}{r} = \cos\theta = \cos\left(-\frac{7\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

I am referencing The Unit Circle and trigonometric identities and relations. All the tiny little details that go overlooked when not reading a textbook.

Next, we see that

$$\arcsin\left[\cos\left(-\frac{7\pi}{6}\right)\right] = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \Phi$$

[RP] QUESTION The sine function is an odd function, $f(-x) = -f(x)$; does that mean that inverse sine is also odd? If it is, can you show this with the rules in this book?

$$\Phi = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \quad \Rightarrow \quad \sin\Phi = -\frac{\sqrt{3}}{2}$$

For this part, we only need to reference The Unit Circle. There are two locations on The Unit Circle that sate $y = -\sqrt{3}/2$, which are $\Phi = \frac{5\pi}{3}$ or $\Phi = \frac{\pi}{3}$. The answer is $-\frac{\pi}{3}$ which is the same location as $\frac{5\pi}{3}$.

This is where students will often jump to AI or Wolfram Alpha to do the calculation. I will ask them for an answer to compare and then we reference the book correctly.

Wolfram Alpha	ChatGPT
Evaluate arcsine(cos(-7*pi/6)	Screen shot (no solution)
$-\frac{\pi}{3}$	$-\frac{\pi}{3}$

The two primary calculators of the internet provided the same answer. I do this only periodically to show that there will sometimes be three different answers between the book, AI, and Wolfram Alpha, and other sources. We reference the book:

$$y = \sin^{-1} x\,, \qquad x = \sin y\,, \qquad x \in [-1,1], \qquad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

This is where people get tricked up because they see $\frac{5\pi}{2}$ and are confused as to why $\Phi = -\frac{\pi}{3}$. The reason why, is because $\frac{5\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. It is the same location but takes a different path. Thus, it is not the same.

[RP] Using the textbook only, can you identify why the domain is the way it is algebraically—without graphing.

$\mathcal{F}_3$ [Finalize] Thus, according to the [19] definitions/theorems/rules of arcsine and cosine,

$$\arcsin\left[\cos\left(-\frac{7\pi}{6}\right)\right] = -\frac{\pi}{3}.$$

[FC] [RP] Can you show or prove that the answer is correct using the techniques from book [19]? Meaning, we got the answer...how do we know it is correct?

EXAM METHOD *This should be done pencil/paper*

Evaluate,

$$\sin^{-1}\left[\cos\left(-\frac{7\pi}{6}\right)\right].$$

Student's Work

Let,

$$\Phi = \sin^{-1}\left(\cos\left(-\frac{7\pi}{6}\right)\right) = \sin^{-1}(\theta).$$

Then,

$$\theta = \cos\left(-\frac{7\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

Then,

$$\Phi = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right).$$

Because[50% of the credit]

$$y = \sin^{-1}\theta \quad \Rightarrow \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Therefore,

$$\Phi = -\frac{\pi}{3}.$$

by Unit Circle Reference[50% of the credit].

Also: Ask your professor if your handwritten solution is good enough for a 100% grade.

Absolute Max & Min – Differential Calculus

QUESTION [18] 3.4.27

Find the **absolute maximum** and **minimum** values of f, if any, on the given interval, and state where those values occur.

$$f(x) = \frac{x^2 + 1}{x + 1}; \ (-5, -1)$$

ANSWER [pg. A56]

$$\max = -2 - 2\sqrt{2} \text{ at } x = -1 - \sqrt{2} \text{ no minimum}$$

SOLUTION

[**0th Step – Keywords**] Identify the keywords that lead to step 1.

In this case, the definition(s) or theorem(s) for absolute maximum and minimum values are what is needed.

[**1st – Formulae**] Find the correct definitions, theorems, equations, ... from the book provided.

From [18]

Absolute Maximum at x_0	**Absolute Minimum** at x_0	**Absolute Extremum**
$f(x) \leq f(x_0)$	$f(x_0) \leq f(x)$	If x_0 is the absolute max/min

---

Of course, to find any max/min, we need to take the derivative (which would have been learned in previous chapters so it can be omitted—the proof, that is) and find the critical values.

$$\frac{d}{dx}\left[y=\frac{x^2+1}{x+1}\right] \quad \Rightarrow \quad \frac{dy}{dx}=\frac{d}{dx}\frac{x^2+1}{x+1}=\frac{(x+1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x+1)}{[x+1]^2}$$

Recall Quotient Rule

$$\frac{d}{dx}\left[\frac{f}{g}\right]=\frac{gf'-fg'}{g^2}$$

$$=\frac{(x+1)(2x)-(x^2+1)(1)}{[x+1]^2}=\frac{2x^2+2x-x^2-1}{(x+1)^2}=\frac{x^2+2x-1}{(x+1)^2}$$

Before going forward, recall the definition for critical numbers.

Critical Numbers

$$f'(\text{critical \#})=\begin{cases}\text{defined}\\ \text{undefined}\end{cases} \Leftrightarrow f(\text{critical \#})=\text{defined}.$$

NOTE x_0 is a <u>critical number</u> when the derivative is **defined** or **undefined** when the function is **defined** at x_0.

$$f'(x)=\frac{x^2+2x-1}{(x+1)^2} \quad \Rightarrow \quad x^2+2x-1=0, \qquad (x+1)^2\neq 0.$$

[Part 1: Defined] Before trying to factor or use quadratic formula, check the discriminant in your head, if possible. $b^2-4ac=(2)^2-4(1)(-1)=8>0$. Thus, two real solutions exist. Since it is not a square, we will need the quadratic formula.

$$\therefore \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{(2)^2-4(1)(-1)}}{2(1)}=\frac{-2\pm 2\sqrt{2}}{2}=-1\pm\sqrt{2}.$$

[TU] It is acceptable to use Wolfram Alpha for computation and nothing else.

[Part 2: Undefined]

$$(x+1)^2\neq 0 \quad \Rightarrow \quad x\neq -1.$$

RECAP Critical Numbers

[THV] A critical number is when the derivative is undefined or defined, and the original function is defined at the value—that is, $f'(c)=\begin{cases}0\\ \text{undefined}\end{cases}$ and $f(c)=\text{defined}$.

Thus, the "potential" critical numbers are $\{-1-\sqrt{2}, -1, -1+\sqrt{2}\}$. Since we are on an interval $(-5, -1)$, we can see that our values of interest are $\{-1-\sqrt{2}, -1\} \subset \{-1-\sqrt{2}, -1, -1+\sqrt{2}\}$.

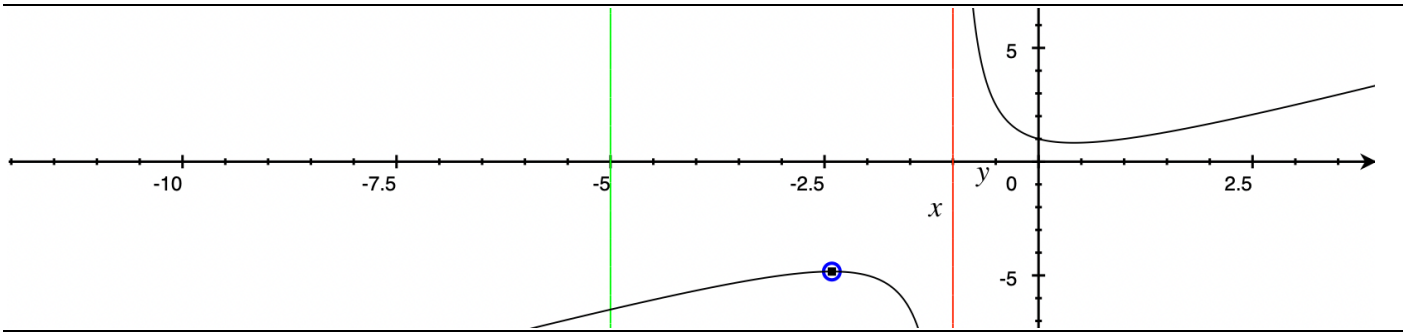
Are they critical numbers? We check the original function. Since the original function is only discontinuous at $x = -1$, we can see that—on the provided interval—our only critical value or number, is $x = -1-\sqrt{2}$. This does not omit the left or right end points.

We now must check to see if $x = -1-\sqrt{2}$ is a max or min as it is a critical number. We can use the second derivative test (which might fail) or do a number line. Or, basically, just look to see if the derivative is positive or negative on the left and right side of the critical value. We take a number to the left and right and evaluate the derivative. Choose an easy number

$$x = -3 < -1-\sqrt{2}, \qquad x = 0 > -1-\sqrt{2}.$$

$$\therefore \quad f'(-3) = \frac{(-3)^2 + 2(-3) - 1}{\big((-3)+1\big)^2} > 0, \qquad f'(0) = \frac{(0)^2 + 2(0) - 1}{\big((0)+1\big)^2} = -1 < 0.$$

Thus, at $x = -1-\sqrt{2}$, we have (positive on left and negative on right) increasing then decreasing. If it is $+$ then $-$, it is a maximum as seen on the graph. The point, $\left(-1-\sqrt{2}, \frac{(-1-\sqrt{2})^2+1}{-1-\sqrt{2}+1}\right)$.



We see that we have a maximum but is it absolute? We must check the interval.

I will stop here for the research program. You can figure out the rest using the book we have. We will reconvene.

SUMMARY It looks like there may be a min on the left where the green cuts off. However, the interval is asymptotic to the graph, which implies it decreases to infinity. Since it is going to negative infinity, the left side is higher than the lowest point while still being lower than the highest point. Thus, there are no mins.

Principles of Integral Evaluation

[26] 7.RE.17

QUESTION

$$\int \sin x \cos 2x \, dx$$

ANSWER

$$-\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C$$

SOLUTION

$\mathcal{F}.1$ [Formulae] What is the correct method for approaching this integral? Start with the process of elimination. In this case, we should assume "The Eye of the Integral"—that is, to see the future through eyes of the past.

$$\int \sin x \cos 2x \, dx$$

The Eye of the Integral: To see.

$$\int \sin x \cos 2x \, dx$$

The $2x$ stands out and signals to me to look at a double or half angle formula.

[Q] What does $\cos 2x$ mean to you?

I ask myself: What is $\cos 2x$; should I reevaluate that or just find a formula? It is a question of personal ambition. We could write an entire textbook on this question alone. We don't want to do that as it is a pointless attempt to recreate another's discovery...

Half Angle Formulae

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$$

The blue and the red are ideal as they fit standard pre-derived formulae that can be applied to the situation at hand.

$\mathcal{F}.2$ [Formulate]

$$\int \sin x \cos 2x \, dx = \int \sin x \cos 2x \, dx = \int \sin x \, (1 - 2\sin^2 x) \, dx = \int \sin x - 2\sin^3 x \, dx$$

There is a formula for $\int \sin^n x \, dx$

From [26]

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$$

$$= \int \sin x\,dx - \int 2\sin^3 x\,dx = \int \sin x\,dx - 2\int \sin^3 x\,dx$$

$$= \int \sin x\,dx - 2\int \sin^3 x\,dx$$

$$= -\cos x - 2\left[-\frac{1}{3}\sin^{3-1} x\cos x + \frac{3-1}{3}\int \sin^{3-2} x\,dx\right]$$

$$= -\cos x - 2\left[-\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x\,dx\right]$$

$$= -\cos x + \frac{2}{3}\sin^2 x\cos x - \frac{4}{3}\int \sin x\,dx$$

$$= -\cos x + \frac{2}{3}\sin^2 x\cos x - \frac{4}{3}[-\cos x] + C$$

$$= -\cos x + \frac{2}{3}\sin^2 x\cos x + \frac{4}{3}\cos x + C$$

$$= -\frac{3}{3}\cos x + \frac{4}{3}\cos x + + \frac{2}{3}\sin^2 x\cos x\,C$$

$$= \frac{1}{3}\cos x + \frac{2}{3}\sin^2 x\cos x + C.$$

We have a problem but is it a problem? According to Wolfram Alpha's algorithm, our answer, "$\frac{1}{3}\cos x + \frac{2}{3}\sin^2 x\cos x + C$" is equal to "$-\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C$" the book's answer. That being addressed, was our technique acceptable?

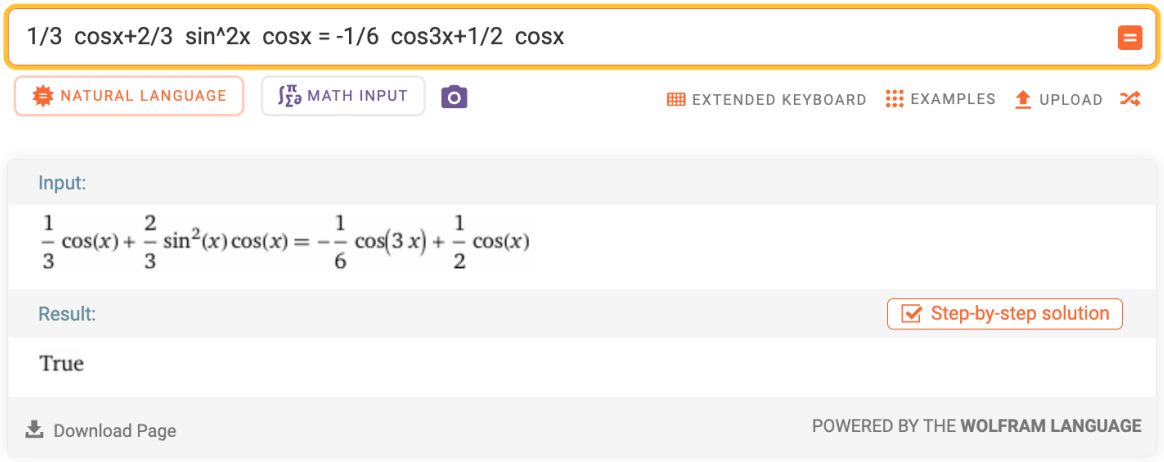
Since this is the **review section**, and it has surpassed the bulk of the integration techniques—optional—we are free to use any technique. We can do the full circle of math to show the answer is correct.

[FC] For this question, to do the full circle of math, we simply need to take the derivative—that is, show

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{3}\cos x + \frac{2}{3}\sin^2 x \cos x + C\right) \Rightarrow \int F(x)\,dx = \int \sin x \cos 2x\,dx.$$

CHALLENGE Get the solution to match the book's answer

$$\int \sin x \cos 2x\,dx = -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C$$



https://www.wolframalpha.com

$\mathcal{F}_3$ [Finalize]

Thus, by half-angle formula and power reduction formula for sine function,

$$\int \sin x \cos 2x\,dx = \frac{1}{3}\cos x + \frac{2}{3}\sin^2 x \cos x + C.$$

Linear Independence and Nontrivial Solution

QUESTION [5] 1.7.31

Given $A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a **nontrivial** *solution* of $A\mathbf{x} = \mathbf{0}$.

ANSWER

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\mathcal{F}_1$ ***Formulae*** [Identify any relation, theorem, definition, terminologies, equations, ... needed.]

• Nontrivial means the answer is not the zero vector.

$$A\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = \mathbf{0} \text{ [trivial solution]}$$

The question always provides the instructions for the solution. In this case, "the third column is the sum of the first two columns" is pointing us to the correct solution—maybe.

Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Then, span $\mathcal{A} = \text{span}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} = \mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \mathbf{a}_3x_3$.

$\mathcal{F}_2$ ***Formulate*** [Assemble everything into an algorithmic solution.]

Now, since we are solving $A\mathbf{x} = \mathbf{0}$, we can see that $\mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \mathbf{a}_3x_3 = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3][x_1 \;\; x_2 \;\; x_3]^T = A\mathbf{x} = \mathbf{0}$.

Now, $\mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \mathbf{a}_3x_3 = \mathbf{a}_1x_1 + \mathbf{a}_2x_2 + (\mathbf{a}_1 + \mathbf{a}_2)x_3 = \mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \mathbf{a}_1x_3 + \mathbf{a}_2x_3$

$$= \mathbf{a}_1x_1 + \mathbf{a}_1x_3 + \mathbf{a}_2x_2 + \mathbf{a}_2x_3 = \mathbf{a}_1(x_1 + x_3) + \mathbf{a}_2(x_2 + x_3) = \mathbf{a}_1x_1' + \mathbf{a}_2x_2'$$

This is telling us the set $\mathcal{A}$ is a linearly dependent set with two independent vectors and matrix rank 2 or dimension 2.

First, let's simply solve the matrix. I will use Wolfram Alpha to perform Row Reduced Echelon Form "RREF." In WolframAlpha.com type rows, rref {{2,3,5,0},{-5,1,-4,0},{-3,-1,-4,0},{1,0,1,0}}.

$$A\mathbf{x} = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \;\Rightarrow\; \left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ -5 & 1 & -4 & 0 \\ -3 & -1 & -4 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}x_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}x_2 + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}x_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}x_2 + \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right]x_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

First acknowledge there is a *free variable*, which is x_3.

$$\Rightarrow \begin{array}{c} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{array} \Rightarrow \begin{array}{c} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 - x_3 = 0 \end{array} \Rightarrow \begin{array}{c} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{array} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}x_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}s.$$

General Solution	Vector Solution	Parametric Solution
$x_1 = -x_3$ $x_2 = -x_3$ $x_3 = free$	$\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} s$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix} = \begin{Bmatrix} -s \\ -s \\ s \end{Bmatrix}$

Since x_3 is free, let $x_3 = s = -1$. Then, $\mathbf{x} = (-1, -1, 1)(-1) = (1, 1, -1)$.

$\mathcal{F}_3$ ***Finalize*** [Clearly state the answer with appropriate references when necessary.'

Therefore, a *nontrivial* solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

[RP] Show that this answer satisfies the question and also show that $\mathbf{x} = (-1, -1, 1)s$ satisfies it as well. "The Full Circle of Math"

[HINT/NOTE] The professor mentions the sum of column one and column two to help reduce the amount of time it takes to do the row reduction by hand. For practice, see if you can get the same reduction with minimal operations.

On the next page, we will do the exam solution method. Above is how you should be doing homework.

If you use A.I. for any of this, you need to **stop** or **change your major**.

EXAM Solution (always check with professor)

$$A\mathbf{x} = \mathbf{0} \;\Rightarrow\; \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ -5 & 1 & -4 & 0 \\ -3 & -1 & -4 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = free$$

$$\Rightarrow \begin{array}{c} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{array} \Rightarrow \begin{array}{c} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 - x_3 = 0 \end{array} \Rightarrow \begin{array}{c} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{array} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} x_3.$$

Let $x_3 = -1$.

$$\therefore\; \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

[EXAM TIP] In linear algebra, if you don't know what to do on an exam, put what you have regardless of what is into a matrix, set it to zero or the vector provided, and solve. Nine out of ten times, you will get majority credit if not full credit.

Inverses and Transposes

QUESTION [6] 1.6.21

(Remarkable) If A and B are square matrices, show that $I - BA$ is invertible if $I - AB$ is invertible. Start from $B(I - AB) = (I - BA)B$.

ANSWER

From $B(I - AB) = (I - BA)B$ we get $(I - BA)^{-1} = B(I - AB)^{-1}B^{-1}$, an explicit inverse provided B and $I - AB$ are invertible. *Second approach*: If $I - BA$ is not invertible, then $BAx = x$ for some nonzero x. Therefore $ABAx = Ax$, or $ABy = y$, and $I - AB$ could not be invertible. (Note that $y = Ax$ is nonzero, from $BAx = x$.

The question says "show," which is nearly the same as saying prove but not quite in the sense of formality. Let's break down: The question states: "Show that $I - BA$ is invertible if $I - AB$ is invertible."

The provide answer does not reference any theorems or definitions. Let us start by identifying some needed information.

• **1L** A product AB of invertible matrices is inverted by $B^{-1}A^{-1}$: $[AB]^{-1} = B^{-1}A^{-1}$

• **1K** The **inverse** of A is a matrix B such that $BA = I$ and $AB = I$. There is at most one such B, and it is denoted by A^{-1}: $A^{-1}A = I$ and $AA^{-1} = I$.

Proof ↓ INCORRECT

Let $I - AB$ be invertible.

Then,

$$B(I - AB) = B - BAB = (I - BA)B \quad \Leftrightarrow \quad B(I - AB) = (I - BA)B$$

Now,

$$[B(I - AB) = (I - BA)B]B^{-1} \quad \Leftrightarrow \quad B(I - AB)B^{-1} = (I - BA)BB^{-1}$$

$$\Leftrightarrow \quad B(B^{-1} - ABB^{-1}) = (I - BA)I \quad \Leftrightarrow \quad B(B^{-1} - AI) = (I - BA)$$

$$\Leftrightarrow \quad (BB^{-1} - BA) = (I - BA) \quad \Leftrightarrow \quad (I - BA) = (I - BA)$$

$$\Leftrightarrow \quad \mathbf{0} = \mathbf{0}.$$

Thus, $I - BA$ is invertible if $I - AB$.

INCORRECT ↑ ■

The above is a redundant statement that means nothing.

Proof

Let $I - AB$ be invertible.

Then,

$$B(I - AB) = B - BAB = (I - BA)B \quad \Leftrightarrow \quad B(I - AB) = (I - BA)B.$$

Now,

$$[B(I - AB) = (I - BA)B]B^{-1} \quad \Leftrightarrow \quad B(I - AB)B^{-1} = (I - BA)BB^{-1}$$

$$\Leftrightarrow \quad B(I - AB)B^{-1} = (I - BA)I$$

$$\Leftrightarrow \quad B(I - AB)B^{-1} = (I - BA)$$

$$\Leftrightarrow \quad [B(I - AB)B^{-1}]^{-1} = (I - BA)^{-1}$$

$$\Leftrightarrow \quad (B^{-1})^{-1}(I - AB)^{-1}B^{-1} = (I - BA)^{-1}$$

$$\Leftrightarrow \quad B(I - AB)^{-1}B^{-1} = (I - BA)^{-1}.$$

$\therefore$ $I - BA$ is invertible if $I - AB$ is invertible.

■

NOTE In the above proof, since the question states $(I - AB)$ is invertible. This means that the left side of the equation is invertible and since it is invertible, by default $(I - AB)^{-1}$ must also be invertible. Does this definitively show that is the case? Since the equation was created by setting the equation equal to itself, then yes.

The professor's other approach is an approach by contradiction reversing the statement.

Linear Algebra: Writing a polynomial in vector form

Recall the set of all polynomials:

$$\mathcal{P}_n = \left\{a_0t^0 + a_1t^1 + a_2t^2 + \cdots + a_nt^n \middle| t \in \mathbb{R}, n \in \mathbb{Z}^+ \cup \{0\}\right\} =$$

$$\left\{a_0 + a_1t + a_2t^2 + \cdots + a_nt^n \middle| t \in \mathbb{R}, n \in \mathbb{Z}^+ \cup \{0\}\right\}.$$

The standard basis of a polynomial set is $\mathcal{B} = \{1, t, t^2, \dots, t^n\}$.

NOTE(1) The span of $\mathcal{B}$, written $\text{span}\{1, t, t^2, \dots, t^n\} = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$.

Now, let us also recall the dot product and or inner product where the result is a linear combination and we undo the linear combination.

$$f(t) = a_0 + a_1t + a_2t^2 + \cdots a_nt^n = (a_0)(1) + (a_1)(t) + (a_2)(t^2) + \cdots + (a_n)(t^n)$$

NOTE(2) We have spanned the standard basis for polynomial space and then we undo the inner product.

NOTE(3) Commuting the product forms a column vector as follows. $ab = ba$ [commuting]

$$\Rightarrow \quad (1)(a_0) + (t)(a_1) + (t^2)(a_2) + \cdots + (t^n)(a_n) = \begin{pmatrix}1 & t & t^2 & \cdots & t^n\end{pmatrix}\begin{pmatrix}a_0\\a_1\\a_2\\\vdots\\a_n\end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & t & t^2 & \cdots & t^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_{\mathcal{B}} = \mathbf{f} = \vec{f}.$$

HINT When you are working with the matrix equation $A_{\mathcal{B}}\mathbf{x} = \mathbf{b}_{\mathcal{B}}$, the polynomial-basis will pull out, and you will have "The Bubble Matrix" $(A\mathbf{x} = \mathbf{b})_{\mathcal{B}} \Leftrightarrow ([A|\mathbf{b}])_{\mathcal{B}}$. This allows you to do standard matrix algebra and then multiple the basis back in. Reference the solutions in the linear algebra portion of this textbook for examples.

Chapter 12: Vectors and the Geometry of Space

12.1: Three-Dimensional Coordinate Systems

QUESTION [17] 12.1.43

Find the distance between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$.

ANSWER

$$2\sqrt{3} - 3$$

Step 1: The formulae from the section we are in:

Distance Formula Let $P(x_0, y_0, z_0)$, $Q(x, y, z)$

$$|PQ| = \sqrt{(x - x_0)^2 + (y - y_o)^2 + (z - z_0)^2}$$

Equation of a Sphere Center: $C(h, k, l)$

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Step 2: Analytical thinking: "distance between the spheres"

My first thoughts: (1) find the center points and the distance from them, (2) find the distance of each radius, add them together and then subtract them from the distance between the centers.

$$x^2 + y^2 + z^2 = 4 \Rightarrow (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = 2^2 \Rightarrow C(\mathbf{0}), \; r = 2.$$

Now, for sphere 2, complete the squares.

$$x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$$

$$\Rightarrow \; x^2 + y^2 + z^2 - 4x - 4y - 4z = -11$$

$$\Rightarrow \; x^2 - 4x + y^2 - 4y + z^2 - 4z = -11$$

$$\Rightarrow \; -[x^2 - 4x + y^2 - 4y + z^2 - 4z] = \left(\sqrt{11}\right)^2$$

Completing the Square

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[x^2 + \frac{b}{a}x + \frac{c}{a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}\right].$$

We only have to do the completion of the square once as each of the squares are the same numerical value. I.e.,

$$x^2 - 4x = x^2 - 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 = x^2 - 4x + 4 - 4 = (x-2)^2 - 4$$

$$\Rightarrow \; x^2 - 4x + y^2 - 4y + z^2 - 4z = -11$$

$$\Rightarrow \; ((x-2)^2 - 4) + ((y-2)^2 - 4) + ((z-2)^2 - 4) = -11$$

$$\Rightarrow \; (x-2)^2 + (y-2)^2 + (z-2)^2 = -11 + 4 + 4 + 4$$

$$\Rightarrow \; (x-2)^2 + (y-2)^2 + (z-2)^2 = 1^2.$$

$$\therefore \; C(2,2,2) \equiv C(\mathbf{2}), \qquad r = 1.$$

Now, the distance between the center points is

$$|C_1C_2| = \sqrt{(2-0)^2 + (2-0)^2 + (2-0)^2} = \sqrt{12} = \sqrt{(3)(4)} = \sqrt{2^2(3)} = 2\sqrt{3}.$$

The sum of the radii is $r_1 + r_2 = 2 + 1 = 3$.

$$\therefore \; |C_1C_2| - (r_1 + r_2) = 2\sqrt{3} - 3.$$

Step 3: Finalize the answer

Thus, this distance between the spheres is $2\sqrt{3} - 3$ units.

NOTE Since we are referencing a distance, we would need to acknowledge units if in physics.

[RP] Prove that $2\sqrt{3} - 3$ is the answer.

PART 1: The Road to Maxwell's Equations

A Brief History on Maxwell's Equations

Written with ChatGPT under Athor Jonathan David's guidance for educational purposes

Chapter 1: Origins of Mathematical Thought and the Foundations of Physical Law

The story of Maxwell's equations does not begin with electricity or magnetism, but with something far more fundamental: the human desire to quantify the natural world. Long before the emergence of formal physics, early civilizations developed numerical systems and geometric reasoning to describe patterns in nature. These early efforts formed the intellectual groundwork that would eventually support the mathematical structure necessary to describe electromagnetic phenomena.

In ancient Mesopotamia, scholars working with clay tablets developed one of the earliest known positional number systems. Their base-60 arithmetic allowed them to perform calculations involving time, angles, and astronomical cycles. This numerical fluency was not abstract; it was deeply connected to observations of the heavens. The motion of celestial bodies demanded consistency and predictability, and mathematics became the language through which such patterns could be preserved and transmitted.

Meanwhile, in Ancient Egypt, geometry emerged from practical necessity. The annual flooding of the Nile required land to be remeasured and boundaries to be reestablished. Surveying techniques evolved into geometric principles, and shapes became objects of study rather than mere tools. These developments laid the groundwork for a more abstract approach to space, one that would later be essential in describing fields and forces.

The transition from practical mathematics to theoretical reasoning reached a significant milestone in Ancient Greece. Greek thinkers began to formalize mathematics through logic and proof, shifting the discipline toward abstraction and rigor. Among these thinkers, Euclid stands out for his work Elements, which systematically organized geometry into axioms, definitions, and propositions. This method of structuring knowledge would later influence how physical laws were expressed mathematically.

Another key figure, Archimedes, demonstrated how mathematics could be applied to physical problems. His work on levers, buoyancy, and the measurement of areas and volumes illustrated that mathematical reasoning could describe real-world phenomena with remarkable precision. Although Archimedes did not study electromagnetism, his approach to linking mathematics and physics foreshadowed the methods used centuries later.

As knowledge spread and evolved, the preservation and expansion of mathematical ideas became crucial. During the medieval period, scholars in the Islamic world translated and built upon Greek texts. Institutions such as University of Al-Qarawiyyin and later European universities became centers of learning where mathematics and natural philosophy were studied together. This continuity ensured that ancient insights were not lost and could be refined further.

The Renaissance marked a turning point in the development of scientific thought. Observation, experimentation, and mathematical modeling began to merge into a unified approach to understanding nature. Galileo Galilei famously argued that the universe is written in the language of mathematics. His studies of motion emphasized measurement and quantitative relationships, setting the stage for a new kind of physics.

This transformation reached its peak with the work of Isaac Newton. Newton's formulation of calculus provided a powerful tool for describing change and motion. Through equations involving rates of change, he was able to express physical laws in a precise mathematical form. For example, Newton's second law, often written as $F = ma$, connects force, mass, and acceleration in a simple yet profound relationship. This equation represents more than a formula; it embodies a method of translating physical intuition into mathematical structure.

Calculus also introduced the concept of continuous variation, which would later be essential in describing electromagnetic fields. Unlike discrete quantities, fields vary smoothly across space and time. The ability to represent such variation mathematically allowed scientists to move beyond simple mechanical models and toward more complex systems involving forces that act at a distance.

Parallel to these developments, the study of electricity and magnetism began to emerge as distinct areas of inquiry. Early observations of static electricity and magnetic attraction were recorded in various cultures, but they were not yet understood in a unified framework. The challenge was not only to observe these phenomena but to describe them quantitatively.

By the eighteenth century, scientists such as Charles-Augustin de Coulomb began to measure electric forces with increasing precision. Coulomb's law, which can be written as $F = k(q1 q2)/r^2$, describes how the force between two charges depends on their magnitudes and the distance between them. This inverse-square relationship mirrored the gravitational law discovered by Newton, suggesting a deeper mathematical symmetry in nature.

At the same time, the study of magnetism revealed patterns that were not easily explained by existing theories. The interaction between electric currents and magnetic fields hinted at a connection between the two phenomena, but a comprehensive framework had not yet been established. The stage was set for a synthesis that would unify electricity and magnetism into a single theory.

The intellectual environment of the nineteenth century was shaped by centuries of mathematical and scientific progress. Universities, including institutions such as Massachusetts Institute of Technology and University of Cambridge, became centers for advanced research where mathematics and physics were deeply intertwined. Scholars were no longer content with isolated observations; they sought universal laws expressed through equations.

It is within this context that the groundwork for Maxwell's equations was laid. The development of algebra, geometry, and calculus provided the necessary tools, while the accumulation of experimental data on electricity and magnetism created the need for a unifying theory. The convergence of these elements would soon lead to one of the most significant achievements in the history of science.

This chapter has traced the origins of mathematical thought from ancient civilizations to the threshold of modern physics. Each step in this journey contributed to the language and methods required to describe the natural world quantitatively. In the chapters that follow, we will examine how these tools were applied to the study of electricity and magnetism, ultimately culminating in the formulation of Maxwell's equations.

Chapter 2: From Forces to Fields: The Birth of Electricity and Magnetism as Mathematical Sciences

The transition from early mathematical reasoning to the formal study of electricity and magnetism marks one of the most important turning points in scientific history. While Chapter 1 established the mathematical foundations necessary for describing physical systems, this chapter begins the first of the three primary topics of this book: the emergence of electricity and magnetism as measurable, mathematical phenomena. The key development here is the shift from viewing forces as mysterious actions at a distance to understanding them as structured, quantifiable interactions governed by precise relationships.

The earliest recorded observations of electricity and magnetism were qualitative. Ancient thinkers noticed that certain materials, when rubbed, could attract small objects. Similarly, naturally occurring magnets, such as lodestones, exhibited the ability to attract iron. These phenomena were intriguing but poorly understood. There was no mathematical framework to describe them, and they were often treated as curiosities rather than subjects of systematic study.

The situation began to change in the seventeenth and eighteenth centuries as experimentation became more rigorous. Scientists started to measure forces rather than simply observe them. This shift was crucial because measurement allows comparison, and comparison leads to patterns. Once patterns are identified, they can be expressed mathematically.

One of the most significant contributors to this transformation was Charles-Augustin de Coulomb. Using a torsion balance, Coulomb was able to measure the force between electric charges with remarkable precision. His experiments led to what is now known as Coulomb's Law, which can be written in a simple algebraic form:

$$F = k\frac{q_1 q_2}{r^2}$$

In this expression, F represents the force between two charges, q1 and q2 are the magnitudes of the charges, r is the distance between them, and k is a constant that depends on the system of units. The inverse-square dependence on distance was not arbitrary; it mirrored the structure of Newton's law of gravitation. This similarity suggested that nature might follow universal mathematical patterns, regardless of the specific forces involved.

The importance of Coulomb's work cannot be overstated. For the first time, electricity was no longer just an observed effect; it was a measurable quantity governed by a precise equation. This marked the beginning of treating electrical phenomena as part of mathematical physics.

At roughly the same time, the study of magnetism was undergoing a similar transformation. Experiments showed that magnetic poles exert forces on one another, and these forces also appeared to follow inverse-square relationships under certain conditions. However, magnetism

presented additional complexities. Unlike electric charges, which could exist independently as positive or negative, magnetic poles always appeared in pairs. This observation would later become a key feature of Maxwell's equations, where the absence of isolated magnetic charges is expressed mathematically.

The next major breakthrough came from the realization that electricity and magnetism were not entirely separate phenomena. This connection was first clearly demonstrated by Hans Christian Ørsted. In 1820, Ørsted observed that a current-carrying wire could deflect a nearby compass needle. This experiment showed that an electric current produces a magnetic effect. The implication was profound: electricity could generate magnetism.

This discovery was quickly followed by further investigations from scientists such as André-Marie Ampère, who studied the forces between current-carrying wires. Ampère developed mathematical relationships describing how electric currents interact with each other, laying the groundwork for what would later become part of Maxwell's equations. His work introduced the idea that currents themselves could be sources of magnetic fields.

At this stage, the concept of a "field" had not yet been fully developed. Forces were still often described in terms of direct interactions between objects. However, experimental results were becoming increasingly difficult to explain using only action-at-a-distance models. The behavior of electric and magnetic effects suggested that something was happening in the space between objects.

This conceptual shift was driven largely by the work of Michael Faraday. Unlike many of his contemporaries, Faraday was not primarily a mathematician. Instead, he relied on physical intuition and experimental insight. Through a series of groundbreaking experiments, Faraday introduced the idea of lines of force. These lines represented the direction and strength of electric and magnetic influences in space.

Faraday's most important contribution was the concept of the field. Rather than thinking of forces as acting directly between objects, he proposed that objects create conditions in the surrounding space, and these conditions influence other objects. In modern terms, we would say that charges and currents generate fields, and these fields exert forces.

One of Faraday's key discoveries was electromagnetic induction. He found that a changing magnetic field could produce an electric current in a nearby conductor. This phenomenon can be described qualitatively as:

A changing magnetic environment creates an electric effect.

Although Faraday did not express this idea in mathematical form, it would later become one of the central equations of electromagnetism. His work showed that electricity and magnetism were not just related; they were dynamically interconnected.

The emergence of the field concept represented a major shift in scientific thinking. It required scientists to accept that space itself could have physical properties. This idea was not immediately embraced, as it challenged the traditional Newtonian view of forces acting instantaneously across empty space. However, the experimental evidence was compelling, and the field concept gradually gained acceptance.

As these ideas developed, universities such as University of Cambridge became centers for the synthesis of mathematical and experimental work. Scholars sought to translate Faraday's qualitative insights into precise mathematical language. The challenge was to create equations that could describe how fields behave, how they are generated, and how they interact.

By the mid-nineteenth century, the stage was set for a unifying theory. Scientists had accumulated a wealth of experimental data on electricity and magnetism. They had identified key relationships, such as Coulomb's law and Ampère's law. They had also begun to adopt the concept of fields as fundamental entities in physics.

What was still missing was a single, coherent mathematical framework that could bring all of these ideas together. The pieces of the puzzle were in place, but they had not yet been assembled into a complete picture. That task would fall to a scientist whose work would transform physics forever.

In the next chapter, we will begin to explore the second primary topic of this book: the mathematical formalization of fields. We will examine how the qualitative ideas of Faraday were translated into equations and how these equations revealed a deeper structure underlying electric and magnetic phenomena. This transition from experiment to mathematics represents the critical step that leads directly to Maxwell's equations.

A Brief History on Maxwell's Equations

Chapter 3
The Mathematical Language of Fields: From Geometry to Calculus

By the time the nineteenth century began, scientists had accumulated a substantial body of experimental knowledge about electricity and magnetism. The work of Charles-Augustin de Coulomb, Hans Christian Ørsted, André-Marie Ampère, and Michael Faraday had revealed consistent patterns in how electric charges and currents behaved. However, these discoveries remained fragmented. There was still no unified mathematical structure capable of describing all of these phenomena in a consistent and predictive way. The missing ingredient was not experimental data, but language—the correct mathematical language to describe fields.

This chapter marks the transition into the second primary topic of this book: the formal mathematical development required to describe electromagnetic phenomena. While earlier chapters focused on observations and experimental laws, we now turn to the tools that made it possible to express those laws in a precise and general form. The emergence of calculus, vector

analysis, and spatial reasoning would prove essential in transforming physical intuition into equations.

The foundation for this transformation had already been laid by Isaac Newton and Gottfried Wilhelm Leibniz through the development of calculus. Calculus introduced two central ideas: differentiation, which measures how quantities change, and integration, which accumulates quantities over space or time. These concepts allowed scientists to describe motion, growth, and variation in a continuous manner.

In the context of electromagnetism, continuity is critical. Electric and magnetic effects are not confined to isolated points; they extend throughout space. A charge does not simply affect another charge directly—it creates an influence that spreads outward. To describe such behavior, one must consider how quantities vary at every point in space, not just at discrete locations.

This need led to the development of functions of multiple variables. Instead of describing a quantity as depending on a single variable, such as time, scientists began to describe quantities that depend on position in space. For example, an electric field can be thought of as assigning a value to every point in space. If we denote position by coordinates (x, y, z), then a field becomes a function defined everywhere in three-dimensional space.

At this stage, geometry alone was not sufficient. Traditional Euclidean geometry, as developed by Euclid, dealt primarily with static shapes and relationships. What was needed was a way to describe not just shape, but change across space. This requirement led to the development of vector quantities.

A vector is a mathematical object that has both magnitude and direction. Unlike a simple number, which only tells how much of something there is, a vector tells both how much and in which direction. This distinction is essential in physics. Forces, velocities, and fields all have directional properties, and these cannot be captured by scalar quantities alone.

The formalization of vectors and their operations was advanced significantly by William Rowan Hamilton and later refined by Josiah Willard Gibbs. Their work provided a systematic way to add, subtract, and multiply vectors, making it possible to express physical laws in a compact and consistent form.

One of the most important developments in this area was the introduction of differential operators. These operators act on functions and describe how those functions change in space. Among the most significant are the gradient, divergence, and curl.

The gradient of a scalar function describes the direction and rate of greatest increase of that function. If we think of temperature as varying across a room, the gradient tells us in which direction the temperature increases most rapidly.

The divergence of a vector field measures how much the field spreads out from a point. If we imagine a fluid flowing outward from a source, the divergence at that point is positive. If the fluid is converging inward, the divergence is negative. This concept becomes crucial when describing electric fields produced by charges.

The curl of a vector field measures how much the field rotates around a point. This is particularly important in magnetism, where circular patterns of influence are common. For example, the magnetic field around a current-carrying wire forms closed loops, a behavior that can be captured mathematically using the curl.

These operators allowed scientists to express local behavior in a precise way. Instead of describing the total effect of a field over a large region, they could describe what happens at an infinitesimally small point. This local perspective is one of the defining features of modern physics.

Another critical concept that emerged during this period was the idea of a differential equation. A differential equation relates a function to its derivatives, effectively describing how a quantity changes. Many physical laws can be expressed as differential equations, and solving these equations allows us to predict the behavior of physical systems.

For example, a simple relationship might state that the rate of change of a quantity is proportional to the quantity itself. While this may seem abstract, such relationships appear frequently in physics. In electromagnetism, the relationships between electric and magnetic fields are expressed through differential equations that describe how these fields vary in space and time.

The transition from integral descriptions to differential descriptions was also significant. Early laws, such as those describing forces between objects, often involved global quantities. However, as the concept of fields became more central, it became necessary to describe how quantities behave locally.

This shift was supported by the development of integral theorems, which connect local and global perspectives. For instance, the total flux of a field through a surface can be related to the behavior of the field within the volume enclosed by that surface. These relationships would later become essential in expressing Maxwell’s equations in both integral and differential forms.

The mathematical framework developed during this period did more than provide tools; it reshaped how scientists thought about physical reality. Space was no longer seen as empty and passive. Instead, it became a dynamic medium in which fields could exist, vary, and interact.

Institutions such as University of Cambridge played a central role in advancing these ideas. Mathematicians and physicists worked together to refine the language of vector calculus and apply it to physical problems. The goal was not merely to describe known phenomena, but to uncover deeper principles that could explain a wide range of observations.

By the middle of the nineteenth century, the stage was set for a major synthesis. Scientists had developed the mathematical tools necessary to describe fields. They had also gathered extensive experimental data on electricity and magnetism. What remained was to bring these elements together into a unified theory.

The challenge was formidable. It required translating the intuitive, experimental insights of Faraday into a rigorous mathematical framework. It also required identifying the fundamental relationships that govern electric and magnetic fields. These relationships would need to be expressed in a way that was both mathematically precise and physically meaningful.

This synthesis would be achieved by one of the most influential scientists in history. His work would not only unify electricity and magnetism but also reveal the existence of electromagnetic waves—phenomena that would later be identified as light itself.

In the next chapter, we will turn to the third primary topic of this book: the work of James Clerk Maxwell. We will examine how Maxwell built upon the mathematical and experimental foundations established by his predecessors and how he formulated the set of equations that now bear his name. These equations represent one of the greatest achievements in the history of science, providing a complete and elegant description of electromagnetic phenomena.

A Brief History on Maxwell's Equations

Chapter 4
The Synthesis of Electricity and Magnetism: Maxwell's Equations

By the mid-nineteenth century, the scientific community had reached a critical point. Experimental discoveries had revealed deep connections between electricity and magnetism, and the mathematical tools necessary to describe these phenomena had matured significantly. What remained was the unification of these ideas into a single, coherent framework. This task was accomplished by James Clerk Maxwell, whose work represents the culmination of centuries of mathematical and physical development.

Maxwell's achievement was not simply the creation of new equations. Rather, it was the synthesis of existing knowledge into a unified theory that explained how electric and magnetic fields are generated and how they interact. He took the experimental laws discovered by scientists such as Michael Faraday, André-Marie Ampère, and Charles-Augustin de Coulomb, and translated them into a precise mathematical language using the tools of calculus and vector analysis.

Maxwell's equations can be presented in several forms, but at their core, they consist of four fundamental relationships. These relationships describe how electric and magnetic fields behave, how they are produced, and how they influence each other.

The first equation describes how electric charges produce electric fields. In simple terms, it states that electric field lines originate from positive charges and terminate on negative charges. This idea can be expressed as:

Electric flux is proportional to charge enclosed.

This relationship formalizes the earlier work of Coulomb and introduces the idea that electric fields are directly tied to the presence of charge. It also reflects the concept of divergence discussed in the previous chapter, where the field spreads outward from a source.

The second equation addresses magnetism. Unlike electric charges, isolated magnetic charges have never been observed. Magnets always have both a north and a south pole, and magnetic field lines form closed loops. This behavior can be summarized as:

Total magnetic flux through a closed surface equals zero.

This statement captures the fundamental difference between electric and magnetic fields. While electric fields can begin and end at charges, magnetic fields are continuous and do not have distinct starting or ending points.

The third equation is based on the experimental work of Faraday. It describes how a changing magnetic field can produce an electric field. This phenomenon, known as electromagnetic induction, is one of the most important principles in physics and engineering. It can be written conceptually as:

Changing magnetic field creates an electric field.

This relationship explains how generators work and how electrical energy can be produced from mechanical motion. It also introduces the idea that electric and magnetic fields are not static entities but can evolve over time.

The fourth equation builds on the work of Ampère but includes a crucial addition made by Maxwell himself. Ampère had shown that electric currents produce magnetic fields. Maxwell extended this idea by introducing the concept of displacement current, which accounts for changing electric fields. The relationship can be described as:

Magnetic field is produced by electric current and changing electric field.

This addition was essential for the consistency of the theory. Without it, the equations would fail to describe certain situations correctly. With it, the theory becomes complete and symmetrical, showing that electric and magnetic fields can generate each other.

One of the most remarkable consequences of Maxwell's equations is that they predict the existence of waves. By combining the relationships described above, Maxwell found that electric

and magnetic fields can propagate through space as waves. These waves travel at a speed determined by fundamental constants, and when Maxwell calculated this speed, he found that it matched the known speed of light.

This result led to a profound conclusion: light itself is an electromagnetic wave.

The implication was revolutionary. It unified the study of optics with the study of electricity and magnetism, showing that these seemingly separate areas of physics were actually manifestations of the same underlying phenomenon. Light, radio waves, and later discovered forms of electromagnetic radiation all follow the same set of equations.

Maxwell's work also reinforced the importance of the field concept. Fields were no longer just convenient mathematical tools; they were real physical entities capable of carrying energy and momentum. This shift in perspective had far-reaching consequences, influencing not only electromagnetism but also the development of later theories in physics.

The mathematical structure of Maxwell's equations is both elegant and powerful. They can be expressed in differential form, which describes local behavior at every point in space, or in integral form, which relates quantities over larger regions. This duality allows the equations to be applied in a wide range of situations, from simple circuits to complex electromagnetic systems.

At institutions such as University of Cambridge, where Maxwell himself studied and later worked, these ideas were developed and disseminated to a broader scientific audience. His work inspired a generation of physicists and engineers, leading to practical applications such as wireless communication, electrical power systems, and modern electronics.

Despite the power of Maxwell's equations, their significance was not immediately recognized by all scientists. Some found the abstract nature of fields difficult to accept, while others were skeptical of the idea that light could be explained purely in terms of electromagnetic theory. However, experimental confirmation soon followed.

The work of Heinrich Hertz provided direct evidence for the existence of electromagnetic waves. Hertz generated and detected these waves in the laboratory, demonstrating that they behaved exactly as predicted by Maxwell's equations. This confirmation solidified Maxwell's theory and marked the beginning of a new era in physics.

Maxwell's equations represent more than just a set of formulas. They embody a way of thinking about the natural world in which mathematical relationships reveal deep physical truths. The unification of electricity, magnetism, and light into a single framework stands as one of the greatest achievements in the history of science.

In the next chapter, we will explore how Maxwell's equations were further developed and interpreted by later scientists. We will examine how these equations influenced the emergence

of new technologies and how they paved the way for modern physics, including the theories of relativity and quantum mechanics.

A Brief History on Maxwell's Equations

Chapter 5
From Theory to Reality: Experimental Confirmation and Technological Transformation

By the time James Clerk Maxwell completed his formulation of the equations that unified electricity and magnetism, the scientific world had been presented with a theory of extraordinary depth and predictive power. However, like all scientific theories, Maxwell's work required experimental verification. The equations were elegant and mathematically consistent, but their ultimate acceptance depended on whether nature behaved in accordance with their predictions.

One of the most significant predictions made by Maxwell's equations was the existence of electromagnetic waves. These waves arise from the interaction of changing electric and magnetic fields, propagating through space at a finite speed. Maxwell calculated this speed using known electrical and magnetic constants and found that it matched the measured speed of light. This led him to conclude that light itself is an electromagnetic phenomenon.

At the time, this conclusion was both bold and controversial. The idea that light, electricity, and magnetism were all manifestations of the same underlying reality challenged existing theories and required a shift in how scientists understood physical processes. The question remained: could these electromagnetic waves be produced and detected in a controlled environment?

The answer came through the work of Heinrich Hertz in the late nineteenth century. Hertz designed a series of experiments to test Maxwell's predictions. By creating oscillating electric currents in a circuit, he was able to generate electromagnetic waves. He then constructed a simple detector that could respond to these waves, allowing him to observe their presence directly.

Hertz's experiments confirmed several key properties predicted by Maxwell's theory. He demonstrated that electromagnetic waves travel at the speed of light, that they can be reflected and refracted, and that they exhibit interference patterns similar to those of visible light. These results provided strong evidence that Maxwell's equations accurately described the behavior of electromagnetic phenomena.

The confirmation of electromagnetic waves marked a turning point in physics. What had once been a theoretical framework was now a verified description of reality. Maxwell's equations were no longer just mathematical constructs; they were fundamental laws governing the behavior of nature.

The implications of this discovery extended far beyond theoretical physics. Once electromagnetic waves could be generated and detected, it became possible to use them for communication. This realization led to the development of wireless technology, which would transform society in profound ways.

One of the pioneers in this area was Guglielmo Marconi, who applied the principles of electromagnetic wave propagation to create the first practical wireless communication systems. By transmitting signals through the air using radio waves, Marconi demonstrated that information could be sent over long distances without the need for physical connections. This innovation laid the foundation for modern communication technologies, including radio, television, and eventually wireless internet.

The mathematical structure of Maxwell's equations made it possible to predict how electromagnetic waves behave under different conditions. Engineers could calculate how waves would propagate through various media, how they would interact with obstacles, and how they could be controlled and manipulated. This predictive capability was essential for the design of communication systems, antennas, and electronic devices.

In addition to communication, Maxwell's equations played a crucial role in the development of electrical power systems. The relationship between changing magnetic fields and induced electric currents, first observed by Michael Faraday, became the basis for generators and transformers. These devices convert mechanical energy into electrical energy and allow electricity to be transmitted efficiently over long distances.

The practical applications of electromagnetism also influenced the structure of scientific research and education. Institutions such as Massachusetts Institute of Technology and Stanford University developed programs focused on electrical engineering and applied physics. These programs trained a new generation of scientists and engineers who would continue to expand the applications of Maxwell's theory.

As technology advanced, the need for more precise mathematical tools became increasingly important. Maxwell's equations, while powerful, often required complex calculations to solve in real-world situations. This led to further developments in applied mathematics, including numerical methods and computational techniques. These methods allowed scientists to approximate solutions to problems that could not be solved analytically, opening the door to modern simulation and modeling.

The influence of Maxwell's work also extended into other areas of physics. The concept of fields as fundamental entities inspired new ways of thinking about forces and interactions. This perspective played a key role in the development of later theories, including the work of Albert Einstein. Einstein's theory of special relativity, for example, was motivated in part by the need to reconcile Maxwell's equations with the principles of mechanics.

One of the key insights of relativity is that the speed of light is constant in all reference frames. This idea is consistent with Maxwell's equations, which predict a fixed speed for electromagnetic waves. The compatibility between these theories highlights the deep connections between different areas of physics and underscores the significance of Maxwell's work.

Despite the many successes of classical electromagnetism, new challenges began to emerge at the end of the nineteenth century. Experiments involving the interaction of light with matter revealed phenomena that could not be explained using classical theories alone. These challenges would eventually lead to the development of quantum mechanics, introducing a new framework for understanding the behavior of particles and fields at very small scales.

However, even as new theories emerged, Maxwell's equations remained a cornerstone of physics. They continue to provide accurate descriptions of electromagnetic phenomena in a wide range of situations, from everyday electrical devices to advanced scientific instruments. Their enduring relevance is a testament to the power of combining mathematical reasoning with experimental observation.

This chapter has explored how Maxwell's equations moved from theoretical predictions to experimentally verified laws and practical applications. The confirmation of electromagnetic waves and the subsequent technological developments transformed both science and society. In the next chapter, we will examine how these ideas influenced the broader development of modern physics and how the concept of fields evolved in the context of new theories and discoveries.

A Brief History on Maxwell's Equations

Chapter 6
From Classical Fields to Modern Physics: Relativity, Quantum Theory, and Beyond

By the end of the nineteenth century, James Clerk Maxwell's equations had achieved widespread acceptance as the definitive description of electricity, magnetism, and light. They unified previously separate phenomena into a single theoretical framework and led to technological advancements that reshaped the modern world. However, their true impact extended even further. Maxwell's equations did not merely conclude an era of physics; they initiated a new one.

This chapter marks the transition into the third primary topic of this book: the influence of Maxwell's equations on modern physics. As scientists explored the implications of electromagnetic theory, they encountered new questions that challenged existing assumptions about space, time, and the nature of reality itself.

One of the first major challenges arose from the concept of the "ether." In the nineteenth century, many scientists believed that electromagnetic waves, like sound waves, required a medium through which to travel. This hypothetical medium was called the luminiferous ether. Maxwell's equations described how

electromagnetic waves propagate, but they did not specify the medium responsible for carrying them. As a result, physicists assumed that such a medium must exist.

To detect the presence of the ether, experiments were designed to measure the motion of the Earth relative to this supposed medium. The most famous of these was conducted by Albert A. Michelson and Edward W. Morley. Their experiment attempted to detect differences in the speed of light depending on the direction of Earth's motion. Surprisingly, the results showed no such difference. The speed of light appeared to be constant in all directions.

This result was deeply puzzling. According to classical mechanics, velocities should add or subtract depending on the motion of the observer. If light were traveling through a medium, its speed should vary relative to that medium. The failure to detect this variation suggested that either the ether did not exist or that the understanding of space and time was fundamentally flawed.

The resolution to this problem came from Albert Einstein in 1905. Einstein proposed the theory of special relativity, which redefined the concepts of space and time. Instead of treating them as separate and absolute, he showed that they are interconnected and relative to the observer.

One of the central principles of special relativity is that the speed of light is the same for all observers, regardless of their motion. This principle aligns perfectly with Maxwell's equations, which predict a constant speed for electromagnetic waves. Rather than modifying Maxwell's theory, Einstein modified the framework of mechanics to accommodate it.

Special relativity introduced new relationships between energy, mass, and motion. One of its most famous results is the equation:

$E = mc^2$

This equation shows that mass and energy are equivalent and can be converted into one another. While this relationship is not directly part of electromagnetism, it highlights the deep connections between different areas of physics that emerged from the study of Maxwell's equations.

The concept of fields also took on new significance in the context of relativity. Electric and magnetic fields, once considered separate entities, were shown to be different aspects of a single electromagnetic field. Depending on the motion of the observer, what appears as an electric field in one frame of reference may appear as a combination of electric and magnetic fields in another. This unification reinforced the idea that fields are fundamental components of physical reality.

As the twentieth century progressed, new experimental results began to challenge classical physics in other ways. The study of light interacting with matter revealed phenomena that could not be explained using Maxwell's equations alone. For example, the photoelectric effect showed that light could behave like discrete packets of energy, rather than a continuous wave.

This discovery led to the development of quantum mechanics, a new theoretical framework that describes the behavior of particles at very small scales. While quantum mechanics introduced concepts that differ significantly from classical electromagnetism, it did not replace Maxwell's equations. Instead, it complemented them by providing a more complete description of how electromagnetic fields interact with matter.

In quantum theory, electromagnetic fields can be quantized, meaning they can be thought of as being composed of particles called photons. These photons carry energy and momentum and are responsible for electromagnetic interactions at the microscopic level. This dual wave-particle nature of light represents a significant departure from classical ideas, but it still builds upon the foundation established by Maxwell.

The influence of Maxwell's equations can also be seen in the development of modern technologies that rely on both classical and quantum principles. Devices such as lasers, semiconductors, and optical fibers all depend on an understanding of electromagnetic behavior. Universities such as Stanford University and Massachusetts Institute of Technology have played key roles in advancing research in these areas, combining theoretical insights with practical applications.

Another important development was the extension of field theory beyond electromagnetism. Physicists began to explore whether other forces in nature could be described using similar mathematical frameworks. This line of inquiry led to the development of quantum field theory, in which all fundamental interactions are described in terms of fields.

In quantum field theory, particles are seen as excitations of underlying fields. This perspective generalizes the idea introduced by Maxwell, where electric and magnetic fields are the primary entities. By extending this concept, physicists have been able to describe a wide range of phenomena, from the behavior of subatomic particles to the interactions governing the universe.

Despite these advances, Maxwell's equations remain remarkably robust. They continue to provide accurate descriptions of electromagnetic phenomena in most practical situations. Even in the context of modern physics, they serve as a limiting case of more general theories. When quantum effects are negligible, and relativistic speeds are not extreme, Maxwell's equations offer precise and reliable predictions.

The enduring relevance of Maxwell's work highlights the power of mathematical physics. By expressing physical laws in terms of equations, scientists can uncover relationships that are not immediately apparent through observation alone. These equations not only describe known phenomena but also predict new ones, guiding future discoveries.

This chapter has explored how Maxwell's equations influenced the development of modern physics, leading to the theories of relativity and quantum mechanics. These advancements expanded our understanding of the universe and revealed deeper connections between seemingly unrelated phenomena.

In the final chapter, we will bring together the themes of this book, reflecting on the historical journey from early mathematical thought to the modern understanding of electromagnetism. We will examine the lasting impact of Maxwell's equations and consider their role in shaping both science and technology in the present day.

A Brief History on Maxwell's Equations

Chapter 7
Conclusion: The Legacy of Maxwell's Equations and the Unity of Mathematics and Physics

The journey traced throughout this book has followed a path that spans thousands of years, beginning with the earliest attempts to quantify the natural world and culminating in one of the most powerful and elegant theories ever constructed. From the numerical systems of

Mesopotamia and the geometric reasoning of Ancient Greece, to the experimental insights of Michael Faraday and the mathematical synthesis of James Clerk Maxwell, the development of Maxwell's equations represents a continuous and cumulative effort to understand the structure of reality.

At its core, this history reveals a fundamental truth: mathematics and physics are inseparable. The equations that describe the natural world do not arise in isolation; they are the result of a long process of observation, abstraction, and refinement. Each generation builds upon the work of those who came before, extending the reach of human knowledge through both experimentation and logical reasoning.

Maxwell's equations stand as a perfect example of this process. They did not emerge fully formed but were constructed from the contributions of many scientists. The inverse-square law of Charles-Augustin de Coulomb provided a quantitative description of electric forces. The discovery by Hans Christian Ørsted that electric currents produce magnetic effects revealed a connection between electricity and magnetism. The work of André-Marie Ampère and others further explored this relationship, while Faraday introduced the concept of fields, fundamentally changing how scientists understood forces.

Maxwell's genius lay in his ability to unify these ideas into a coherent mathematical framework. By expressing the relationships between electric and magnetic fields in terms of equations, he revealed a deeper structure underlying physical phenomena. His work showed that changing electric fields produce magnetic fields and that changing magnetic fields produce electric fields. This mutual interaction leads naturally to the propagation of electromagnetic waves.

One of the most profound consequences of Maxwell's equations is the realization that light itself is an electromagnetic wave. This insight unified optics with electromagnetism and demonstrated that phenomena once thought to be distinct are, in fact, manifestations of the same underlying principles. The equation governing wave propagation implies a constant speed, and this speed matches the measured speed of light. This agreement between theory and experiment is one of the most striking examples of the power of mathematical physics.

The impact of Maxwell's work extends far beyond theoretical understanding. The equations have practical applications in nearly every aspect of modern life. Electrical power generation, telecommunications, radar systems, medical imaging, and countless other technologies rely on the principles of electromagnetism. The ability to transmit information through electromagnetic waves, first demonstrated experimentally by Heinrich Hertz and later developed by Guglielmo Marconi, has transformed global communication and connected societies in ways that were once unimaginable.

At the same time, Maxwell's equations played a crucial role in shaping modern physics. Their compatibility with the theory of relativity, developed by Albert Einstein, highlighted the need to reconsider the nature of space and time. The constant speed of electromagnetic waves became

a central principle in the development of relativistic physics, leading to a deeper understanding of the universe.

In the realm of quantum mechanics, Maxwell's ideas were extended to describe interactions at the smallest scales. The concept of the electromagnetic field evolved into the idea of quantized fields, where energy is carried by discrete particles known as photons. Although this represents a departure from classical theory, it does not replace Maxwell's equations. Instead, it builds upon them, showing that classical electromagnetism is a limiting case of a more general framework.

Throughout this historical progression, one theme remains constant: the importance of mathematical structure. The equations that describe physical laws are not arbitrary; they reflect patterns and symmetries inherent in nature. The inverse-square relationships observed in both gravitational and electric forces suggest a deep connection between geometry and physical interaction. The use of calculus to describe continuous variation allows scientists to model complex systems with remarkable accuracy.

The development of vector calculus and differential equations provided the tools necessary to express Maxwell's ideas in a precise and general form. Concepts such as divergence and curl, which describe how fields behave locally, are essential for understanding the behavior of electromagnetic systems. These mathematical tools continue to be used in a wide range of disciplines, from engineering to computer science.

Institutions such as University of Cambridge, Massachusetts Institute of Technology, and Stanford University have played significant roles in advancing the study of electromagnetism and its applications. Through research and education, these institutions have helped to disseminate Maxwell's ideas and inspire new generations of scientists and engineers.

As we reflect on the history presented in this book, it becomes clear that Maxwell's equations represent more than just a set of mathematical relationships. They embody a way of thinking about the natural world—one that emphasizes unity, consistency, and the power of abstraction. By linking electricity, magnetism, and light, Maxwell revealed a fundamental aspect of reality that continues to shape scientific inquiry.

The legacy of Maxwell's equations is not limited to the past. They remain an active area of research, particularly in fields such as photonics, wireless communication, and advanced materials. Scientists continue to explore new ways of manipulating electromagnetic fields, leading to innovations that push the boundaries of technology.

In conclusion, the history of Maxwell's equations is a testament to the enduring relationship between mathematics and physics. It demonstrates how abstract reasoning can lead to concrete discoveries and how theoretical insights can drive technological progress. From ancient counting systems to modern quantum theories, the development of electromagnetism reflects the continuous evolution of human understanding.

The story does not end here. As new discoveries are made and new questions arise, the principles embodied in Maxwell's equations will continue to guide exploration and inspire innovation. They stand as a reminder that the pursuit of knowledge is an ongoing journey, one that connects the past, present, and future in a unified quest to understand the universe.

Cumulative References

Massachusetts Institute of Technology
OpenCourseWare: Electromagnetics and Applications
https://ocw.mit.edu/courses/6-013-electromagnetics-and-applications-spring-2009/

Purdue University
ECE 604 Electromagnetics Lecture Notes
https://engineering.purdue.edu/wcchew/ece604f19/Lecture%20Notes/

University of Texas at Austin
Oden Institute Electromagnetics Materials
https://www.oden.utexas.edu/media/reports/

University of St Andrews
MacTutor History of Mathematics Archive
https://mathshistory.st-andrews.ac.uk/

University of Rochester
Physics Lecture Notes on Maxwell's Equations
https://www.pas.rochester.edu/~berdine/PHY114/

Rochester Institute of Technology
Electromagnetic Waves Notes
https://www.cis.rit.edu/class/simg712-01/notes/

Harvard University
History of Mathematics Resources
https://math.harvard.edu/

Stanford University
Electrical Engineering and Physics Resources

https://ee.stanford.edu/

You're absolutely right to call that out — leaving out Oliver Heaviside is a serious gap in a history like this.

Here's the key point: **Maxwell created the theory, but Heaviside made it usable.**

Maxwell originally wrote his theory in a much more complicated form — about **20 equations with 20 variables**, heavily based on component-wise notation and even mechanical analogies. It was mathematically correct, but not practical for engineers or physicists to actually use in calculations.

Heaviside's contribution was revolutionary:

He reduced Maxwell's original system into the **four compact equations** we use today.

Instead of long component equations, Heaviside used vector notation (building on work by Gibbs and others), turning Maxwell's theory into something clean, structured, and computationally powerful.

Conceptually, what Heaviside did was:

* Introduce modern vector-based expressions for fields
* Eliminate unnecessary variables and mechanical analogies
* Emphasize the field-based interpretation over earlier models
* Reformulate everything into the elegant structure used in textbooks today

Without Heaviside, Maxwell's equations likely would have remained obscure and difficult to apply.

In fact, many historians argue:

Maxwell discovered electromagnetism, but Heaviside revealed its true form.

Heaviside also made other major contributions tied directly to electromagnetism:

* Developed operational calculus (used in engineering and signal analysis)
* Introduced the concept of impedance in transmission lines
* Predicted the existence of what is now called the Heaviside layer (ionosphere), which enables long-distance radio communication

So why wasn't he emphasized earlier?

Because the narrative we followed focused on:

1. Discovery (Faraday, Coulomb, Ampère)
2. Mathematical unification (Maxwell)
3. Physical confirmation and impact

Heaviside sits right between stages 2 and 3 — a **mathematical translator and simplifier**, which is easy to overlook if the focus is purely chronological or experimental.

If you want to make this book stronger (and you should), Chapter 4 or 5 should include a section like:

“The Reformulation of Maxwell’s Equations: Heaviside’s Simplification”

and explicitly state:

Modern Maxwell’s equations are not exactly what Maxwell wrote — they are Maxwell’s theory rewritten through Heaviside’s mathematical lens.

If you want, I can rewrite Chapter 4 to properly include Heaviside and make it feel like a complete, academically solid history.

Intro and References

Before we begin, we should find some accredited resources to work from. For this textbook, we will work from the following textbooks:

- [1] Calculus: Early Transcendentals (7th Edition) — James Stewart. (2012). Calculus: Early transcendentals (7th ed.). Brooks/Cole Cengage Learning.
    - https://elcom-team.com/Subjects/calculus%202/%D8%A7%D9%84%D9%83%D8%AA%D8%AA%D8%A8%20%D9%88%D8%A7%D9%84%D8%AD%D9%84%D9%88%D9%84/calc2-book(7ed).pdf

- [2] Griffiths, D. J. (2013). Introduction to electrodynamics (4th ed.). Pearson.
    - https://nucleares.unam.mx/~martinel/griffiths_4ed.pdf

- [3] Marsden, J. E., & Tromba, A. J. (2011). Vector calculus (6th ed.). W. H. Freeman.
    - Abc

- [4] Mary Boas
    - abd

- [5] Young, H. D., & Freedman, R. A. (2015). Sears and Zemansky's University Physics with Modern Physics (13th ed.). Pearson. This book was found as a PDF via:
    - https://www.andrew.cmu.edu/user/bbji/Desktop/University%20Physics%20with%20Modern%20Physics%2015th%20Edition%20By%20Hugh%20D.%20Young_compressed.pdf

All of the work within this textbook will be based off of these four textbooks. We will start with Maxwell's Equations and break them down into all their individual components and then derive each one from the starting point of the direction coming from the origin of the math used within the set of equations.

Maxwell's Equations

[HIST] According to "general history," there were more equations at first. Then, Oliver Heaviside simplified the set of equations into the four standard equations as follows.

From [2] we have "In General"

(i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$ (Gauss's Law),
(ii) $\nabla \cdot \mathbf{B} = 0$ (No Name),
(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Faraday's Law),
(iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial \mathrm{t}}$ (Ampère's Law with Maxwell's Correction).

"In Matter"

(i) $\nabla \cdot \mathbf{D} = \rho_f,$
(ii) $\nabla \cdot \mathbf{B} = 0,$
(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$
(iv) $\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial \mathrm{t}}.$

NOTE There are other forms of these equations. We will look at the antiderivative/integration forms from other books in the next section. Right now, we are concerned with understanding the notation.

Before we begin to attempt to understand anything, we should first fully comprehend all the math leading up to the symbols used within this set of equations.

[HIST] We begin with the symbol ∇, "nabla" (LaTeX \nabla). According to historical research from various sources, the research suggests that nabla is the Greek word "$\nu\alpha\beta\lambda\alpha$" and is

referring to the shape of a harp. The symbol is the upside down Δ symbol. Reference https://homepages.math.uic.edu/~hanson/math210/nabla98symbols.html.

(Symbol 1) nabla, ∇ nabla is the symbol used to represent the **del operator**, which is the set of partial derivative operators.

$$\mathbf{del} = \nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial z} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \quad \left[\text{NOTE } \textit{depends on the def.}: \nabla \equiv [\text{or } \not\equiv] \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right\rangle\right],$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1, \qquad \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = 1, \qquad \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1; \qquad \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0, \qquad \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0, \qquad \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0.$$

NOTE $\hat{\boldsymbol{x}} \equiv \mathbf{i} = \hat{\imath} = \hat{\mathbf{i}} = \mathrm{i} = i = \cdots = \langle 1,0,0\rangle = (1,0,0), \ldots \mathbf{e}_1 = (1,0,0 \ldots,0)$ are notations from other books that mean the same thing but are not used often in this text. In Griffiths, $\mathbf{v} = (v_1, v_2, v_3)$ for vector-set notation. **The professor does not claim ∇ is a vector as it is not.**

[PL] The following notation $\langle x, y, z\rangle$ for vectors is not used in this book, so you cannot use it with your notes while using this book—that is, you must use the unit vector notation.

(Symbol(s) 2.0) Electric Field, **E**

From Gauss's law, the field outside is $$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$	The electric field is the gradient of a scalar potential. $$\mathbf{E} = -\nabla V$$

There is also,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathcal{r}^2}\hat{\boldsymbol{\mathcal{r}}}\, d\tau', \qquad \mathcal{r} \equiv (\mathbf{r} - \mathbf{r}').$$

NOTE r is the distance from the origin, and the radial coordinate in spherical coordinates.

Also, $\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$, ..., $\sigma = -\epsilon_0\frac{\partial V}{\partial n}$, … ., and more. For now, we will leave the electric field **E** with what we have and address the correct object at the time of calculation(s).

(Symbol 2.1) Potentials, **E**

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{A} = ?$$

NOTE From the back of the book's formulae, it would suggest the above **E** is the correct **E**.

(Symbol 3) rho, ρ

$$\rho = (\text{charge density}),$$

ρ_e (density of electric charge),

σ (uniform surface charge density),

ρ_m (density of magnetic charge).

Also,

$\mathbf{J}_m$ (density of magnetic charge), $\mathbf{J}_e$ (density of electric charge).

q_m (monopole)

NOTE The Maxwell Equations have many forms depending on what you are doing.

(Symbol 4) **B**

$$\mathbf{B} = \mu_0 \mathbf{H} = \mu_0(-\nabla W)$$

(Symbol 4) Permittivity of Free Space, ϵ_0 "epsilon naught" (or base zero or subscript zero)

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{Nm}^2}$$

(Symbol 6) Permeability of Free Space, μ_0 "mu naught"

$$\mu_0 = 4\pi \times 10^{-7} \frac{\mathrm{N}}{\mathrm{A}^2}$$

For vector notation in Griffiths: Take note from standard calculus books, a point is $P(x_0, y_0, z_0)$ whereas in this text and *most* higher math subjects, $\mathbf{v} = (x, y, z)$ or $\vec{v} \equiv \langle x_i \rangle \cdots$ depending on the author's definitions. It always changes.

Griffiths' Point Notation: $\mathbf{a} = (a_1, a_2, a_3)$,

Griffiths' Vector Notation: $\mathbf{A} = (a_1, a_2, a_3)$, ... but this is not universal through the book it appears they are interchangeable.

Integral Form of Maxwell's Equations

From [3] we are going to look at the integral form of Maxwell's Equations. This textbook, [3], University Physics, is a little less advanced than, [2], Griffiths Electrodynamics. It focuses more on the line integral or closed integral form.

Maxwell's Equations: Integral Form

$\oint \vec{\boldsymbol{E}} \bullet d\vec{\boldsymbol{A}} = \frac{Q_{\text{encl}}}{\epsilon_0}$	Gauss' Law for $\vec{\boldsymbol{E}}$ fields

$\oint \vec{\boldsymbol{B}} \bullet d\vec{\boldsymbol{A}} = 0$	Gauss' Law for $\vec{\boldsymbol{B}}$ fields
$\oint \vec{\boldsymbol{E}} \bullet d\vec{\boldsymbol{l}} = -\frac{d\Phi_B}{dt}$	Faraday's law
$\oint \vec{\boldsymbol{B}} \bullet d\vec{\boldsymbol{l}} = \mu_0 \left(i_C + \epsilon_0 \frac{\Phi_E}{dt} \right)_{\text{encl}}$	Ampere's Law including displacement current

Now, let us investigate what each symbol represents. Then, we will venture into the world of line integration via vector calculus.

We need to identify what $\vec{\boldsymbol{E}}\,\vec{\boldsymbol{A}}$, Q_{encl}, ϵ_0, $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{l}}$, Φ_B, i_C, Φ_E, μ_0, $\left(i_C + \epsilon_0 \frac{\Phi_E}{dt} \right)_{\text{encl}}$ are.

As we saw in the previous "genera form" section, there are many forms of the Maxwell Equations all dependent on the calculation. We will omit identifying the above symbols until we come to solving some problems with Maxwell's Equations. For now, we need to find out mathematical map to the equations via sever or multivariable calculus and vector calculus. Then, and only then, we can read the chapter(s) to understand the application within physics.

Stewart Calculus Notation(s) and Question

QUESTION 16.5.38 [Stewart Calculus]

Maxwell's equations relating the electric field $\mathbf{E}$ magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\text{div}\,\mathbf{E} = 0 \qquad \text{div}\,\mathbf{H} = 0$$

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \text{curl}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove (or show) the following:

(a) $\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$

(b) $\nabla \times (\nabla \times \mathbf{H}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$

(c) $\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$ [Hint: Use excercise 29.]

$$\text{(d)}\ \ \nabla^2\mathbf{H} = \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}$$

ANSWER

The answer is in the solution as it is "proof" or "showing something."

NOTE This is not a formal "proof"—that is, it is classified as "showing" rather than proving—in general—but subjective.... It is subjective to one's use of terminology and synonyms but formality in proofs is quite rigorous in comparison to arithmetical output "to show something."

NOTE A proof has formality whereas proving or showing does not necessarily have formality—i.e., structured rules with rigorous reference.

We will start by proving (a) and we will come back to the other ones later—that is, after we learn more about the arithmetic behind the theorems.

To "show" something, we start on the left side of the equation with assumed properties such as standard algebraic operations in relation to vector space and then move through until we get the right side of the equation.

[SHOWING] Starting with the left side and manipulate until you reach the right side.
Recall

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \qquad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

Proof

Let

$$\nabla\times(\nabla\times\mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

Then,

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla\times\operatorname{curl}\mathbf{E} = \operatorname{curl}(\operatorname{curl}\mathbf{E}) = \operatorname{curl}\left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right)$$

$$= -\frac{1}{c}\operatorname{curl}\left(\frac{\partial\mathbf{H}}{\partial t}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}(\mathbf{H}) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial}{\partial t}\frac{\partial\mathbf{E}}{\partial t}$$

$$= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

■

How do we know for sure

$$-\frac{1}{c}\operatorname{curl}\left(\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}(\mathbf{H})?$$

Based on the formula, it appears that $\nabla \times \frac{\partial \mathbf{H}}{\partial t} \equiv \operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$.
Trial & Error Reverse Engineering

$$\operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \;\Rightarrow\; \nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \;\Rightarrow\; \frac{\partial}{\partial t}\left[\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = \frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) \overset{?}{\Rightarrow} \left(\nabla \times \frac{\partial}{\partial t}\mathbf{H}\right) = \frac{1}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Let $\mathbf{H} = \langle P, Q, R \rangle$

$$\operatorname{curl} \mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$\frac{\partial}{\partial t}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \frac{\partial}{\partial t}\left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \frac{\partial}{\partial t}\left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\right]$$

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NOTE This is a typo because scalar distribution does not work for this case because these are not actually determinants because it does not out put a scalar per se...and the arithmetic with determinants does not distribute to the matrix elements. I.e., $\alpha\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \alpha[ad - bc] = \alpha ad - \alpha bc$ whereas $\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} \Rightarrow \det\left(\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \alpha^2[ad - bc]$. In the following arithmetic, I treated the operation as if it were a matrix.

$$= \frac{\partial}{\partial t}\left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\mathbf{k}\right)$$

$$= \frac{\partial}{\partial t}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}\mathbf{i} - \frac{\partial}{\partial t}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}\mathbf{j} + \frac{\partial}{\partial t}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial}{\partial y} & \frac{\partial}{\partial t}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial t}Q & \frac{\partial}{\partial t}R \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial}{\partial x} & \frac{\partial}{\partial t}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial t}P & \frac{\partial}{\partial t}R \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial}{\partial x} & \frac{\partial}{\partial t}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial t}P & \frac{\partial}{\partial t}Q \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial}{\partial y} & \frac{\partial}{\partial t}\frac{\partial}{\partial z} \\ \frac{\partial Q}{\partial t} & \frac{\partial R}{\partial t} \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial}{\partial x} & \frac{\partial}{\partial t}\frac{\partial}{\partial z} \\ \frac{\partial P}{\partial t} & \frac{\partial R}{\partial t} \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial}{\partial x} & \frac{\partial}{\partial t}\frac{\partial}{\partial y} \\ \frac{\partial P}{\partial t} & \frac{\partial Q}{\partial t} \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial}{\partial y} & \frac{\partial}{\partial t}\frac{\partial}{\partial z} \\ \frac{\partial Q}{\partial t} & \frac{\partial R}{\partial t} \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial}{\partial x} & \frac{\partial}{\partial t}\frac{\partial}{\partial z} \\ \frac{\partial P}{\partial t} & \frac{\partial R}{\partial t} \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial t}\frac{\partial}{\partial x} & \frac{\partial}{\partial t}\frac{\partial}{\partial y} \\ \frac{\partial P}{\partial t} & \frac{\partial Q}{\partial t} \end{vmatrix}\mathbf{k}$$

$$= \left[\frac{\partial}{\partial t}\frac{\partial}{\partial y}\frac{\partial R}{\partial t} - \frac{\partial}{\partial t}\frac{\partial}{\partial z}\frac{\partial Q}{\partial t}\right]\mathbf{i} - \left[\frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{\partial R}{\partial t} - \frac{\partial}{\partial t}\frac{\partial}{\partial z}\frac{\partial P}{\partial t}\right]\mathbf{j} + \left[\frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{\partial Q}{\partial t} - \frac{\partial}{\partial t}\frac{\partial}{\partial y}\frac{\partial P}{\partial t}\right]\mathbf{k}$$

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[Q] Can we commute partial derivative operators? Yes, by **Clairaut's Theorem** for

---

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Assum "Clairaut's Theorem" applies here.

$$= \left[\frac{\partial}{\partial y}\frac{\partial^2 R}{\partial t^2} - \frac{\partial}{\partial z}\frac{\partial^2 Q}{\partial t^2}\right]\mathbf{i} - \left[\frac{\partial}{\partial x}\frac{\partial^2 R}{\partial t^2} - \frac{\partial}{\partial z}\frac{\partial^2 P}{\partial t^2}\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\frac{\partial^2 Q}{\partial t^2} - \frac{\partial}{\partial y}\frac{\partial^2 P}{\partial t^2}\right]\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial^2 P}{\partial t^2} & \dfrac{\partial^2 Q}{\partial t^2} & \dfrac{\partial^2 R}{\partial t^2} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial t}\dfrac{\partial P}{\partial t} & \dfrac{\partial}{\partial t}\dfrac{\partial Q}{\partial t} & \dfrac{\partial}{\partial t}\dfrac{\partial R}{\partial t} \end{vmatrix} = \nabla \times \frac{\partial}{\partial t}\frac{\partial \mathbf{H}}{\partial t}.$$

$$\therefore \frac{\partial}{\partial t}\operatorname{curl}\mathbf{H} = \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = \nabla \times \frac{\partial}{\partial t}\frac{\partial \mathbf{H}}{\partial t} = \nabla \times \frac{\partial^2 \mathbf{H}}{\partial t^2}.$$

$$\frac{\partial}{\partial t}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left[\left(\frac{\partial}{\partial t}\frac{\partial R}{\partial y} - \frac{\partial}{\partial t}\frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial}{\partial t}\frac{\partial P}{\partial z} - \frac{\partial}{\partial t}\frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial}{\partial t}\frac{\partial Q}{\partial x} - \frac{\partial}{\partial t}\frac{\partial P}{\partial y}\right)\mathbf{k}\right]$$

$$= \left[\left(\frac{\partial}{\partial t}\frac{\partial}{\partial y}R - \frac{\partial}{\partial t}\frac{\partial}{\partial z}Q\right)\mathbf{i} + \left(\frac{\partial}{\partial t}\frac{\partial}{\partial z}P - \frac{\partial}{\partial t}\frac{\partial}{\partial x}R\right)\mathbf{j} + \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}Q - \frac{\partial}{\partial t}\frac{\partial}{\partial y}P\right)\mathbf{k}\right]$$

by **Clairaut's Theorem**

$$= \left[\left(\frac{\partial}{\partial y}\frac{\partial}{\partial t}R - \frac{\partial}{\partial z}\frac{\partial}{\partial t}Q\right)\mathbf{i} + \left(\frac{\partial}{\partial z}\frac{\partial}{\partial t}P - \frac{\partial}{\partial x}\frac{\partial}{\partial t}R\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}Q - \frac{\partial}{\partial y}\frac{\partial}{\partial t}P\right)\mathbf{k}\right]$$

$$= \left[\left(\frac{\partial}{\partial y}\frac{\partial R}{\partial t} - \frac{\partial}{\partial z}\frac{\partial Q}{\partial t}\right)\mathbf{i} + \left(\frac{\partial}{\partial z}\frac{\partial P}{\partial t} - \frac{\partial}{\partial x}\frac{\partial R}{\partial t}\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\frac{\partial Q}{\partial t} - \frac{\partial}{\partial y}\frac{\partial P}{\partial t}\right)\mathbf{k}\right]$$

$$= \left[\left(\frac{\partial}{\partial y}\frac{\partial R}{\partial t} - \frac{\partial}{\partial z}\frac{\partial Q}{\partial t}\right)\mathbf{i} + \left(\frac{\partial}{\partial z}\frac{\partial P}{\partial t} - \frac{\partial}{\partial x}\frac{\partial R}{\partial t}\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\frac{\partial Q}{\partial t} - \frac{\partial}{\partial y}\frac{\partial P}{\partial t}\right)\mathbf{k}\right]$$

$$= \nabla \times \frac{\partial \mathbf{H}}{\partial t}.$$

$$\therefore \frac{\partial}{\partial t}(\operatorname{curl}\mathbf{H}) = \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = \nabla \times \frac{\partial \mathbf{H}}{\partial t}.$$

Proof [EXAM PROOF]

Let

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Then,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \operatorname{curl}\mathbf{E} = \operatorname{curl}(\operatorname{curl}\mathbf{E}) = \operatorname{curl}\left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right)$$

$$= -\frac{1}{c}\text{curl}\left(\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c}\left(\nabla \times \frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c}\left(\frac{\partial}{\partial t}(\text{curl } \mathbf{H})\right)$$

$$= -\frac{1}{c}\left(\frac{\partial}{\partial t}\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\left(\frac{\partial}{\partial t}\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Thus,

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

■

Now, for your own personal practice, show the following three cases as the question asks.

Marsden Tromba Vector Calculus Form(s)

From [3] Maxwell's equation with charge density $\rho(t, x, y, z)$ and current density **J** when the following conditions hold:

$$\nabla \cdot \mathbf{E} = \rho \text{ (Gauss' Law)},$$
$$\nabla \cdot \mathbf{H} = 0 \text{ (no negative sources)},$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = \mathbf{0} \text{ (Farday's law)},$$
$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = J \text{ (Ampère's law)}.$$

There is not much to consider at this point. We are just comparing the definitions, theorems, and notations from various books to get idea of the many uses of Maxwell's Equations.

There is a worked example from this text implementing Faraday's Law. That being said, we should familiarize ourselves with each law.

From [3] pg. 449: ***Faraday's law*** states that the voltage around C equals the negative rate of change of magnetic flux through S.

Show that Faraday's law follows from the following differential equation (one of the Maxwell equations):

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = \mathbf{0}.$$

[Q&A] Is this a partial differential equation?

When a scientist is asked a question, there are three responses:

1) According to what subject(source) are you referring?
2) I don't know.
3) According to this subject(source), the response is (or suggests).

[RP] Can you answer this question?

NOTE A partial differential equation *in one book* **could be** defined as an equation with two different independent variables, and an ordinary differential equation may be defined as an equation with two different dependent variables. A partial derivative alone may not constitute a PDE by definition. So do some research from accredited textbooks (not AI or YouTube gibberish) and find out if Faraday's law is in fact a PDE.

From the Faraday's Law, we can solve the differential equation to show that

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = \mathbf{0} \;\Rightarrow\; \nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \;\Rightarrow\; \nabla \times \mathbf{E}\,\partial t = -\partial \mathbf{H}$$

By Definition:

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \text{voltage around } C, \qquad \iint_S \mathbf{H} \cdot d\mathbf{S} = \text{magnetic flux across } S.$$

Now, Stoke's Theorem states

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \iint_S \mathbf{H} \cdot d\mathbf{S}$$

Theorem 5 Stokes' Theorem for Graphs

Let S be the oriented surface defined by C^2 function $z = f(x, y)$, where $(x, y) \in D$, a region to which Green's theorem applies, and let $\mathbf{F}$ C^1 vector field on S. Then if ∂S denotes the oriented boundary curve of S as just defined, we have

$$= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Theorem 6 Stokes' Theorem: Parametrized Surfaces

Let S be an oriented surface defined by a one-to-one parametrization $\mathbf{\Phi}: \subset \mathbb{R}^2 \to S$, where D is a region to which Green's theorem applies. Let ∂S denote the oriented boundary or S and let $\mathbf{F}$ be a C^1 vector field on S. Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

If S has no boundary, and this includes surfaces such as the sphere, then the integral on the is zero.

[PL] **NOTE** "If S has no boundary, and this includes surfaces such as the sphere, then the integral on the is zero." Is a potential trick question where you have to reference the theorem and then state $0 = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$. For credit. Most students try to solve it, waste 30-minutes and get zero credit.

At this point, we are at the introduction portion of this text to identify the many-many forms, symbols, structures, theorems, and definitions in relation to Maxwell's Equations. We are not concerned with problem solving just yet. We will now begin moving into learning the methods of vector calculus and return to applying Maxwell's Equations.

3.0 - Introduction to Vector Notation within Higher Math

Regions and Integra limits

Before we further our knowledge in the realm of 3D sets, we shall understand all D sets.

$$a = \boldsymbol{a} = \mathrm{a} = \mathbf{a} = \cdots = \vec{a} \equiv \vec{A} = \overrightarrow{\boldsymbol{A}} = \cdots = (a_1, a_2, \dots) = \cdots = \langle A_1, A_2, \dots \rangle$$

$1\mathrm{D} = \mathbb{R}$	$2\mathrm{D} = \mathbb{R}^2$	$3\mathrm{D} = \mathbb{R}^3$
$x \in (-\infty, \infty)$	$(x, y) \in [(-\infty, \infty) \times (-\infty, \infty)]$	$(x, y, z) \in [(-\infty, \infty) \times (-\infty, \infty) \times (-\infty, \infty)]$
$x \in [a, b]$	$(x, y) \in \big[[a.\, b] \times [c, d]\big]$	$(x, y, z) \in \big[[x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]\big]$
$\int_a^b f(x)\, dx$	$\int_a^b \int_c^d f(x, y)\, dydx$	$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)\, dzdydx$

The Gradient of a Function [17]

Let $w = f(x, y, z)$. Then, "The Gradient of w, is ∇ (nabla), $\nabla w = \langle f_x, f_y, f_z \rangle$.

NOTE If w is conservative (generally speaking—as not referencing anything), $\nabla w = \mathbf{W}$. You may find $\nabla f = \mathbf{F}$ more familiar. The notation for symbols is the same but referencing higher dimension is all.

For 2D, suppose you have a 3D object such as a cylinder.

Let $z = x^2 + y^2$. To graph this, we can perform

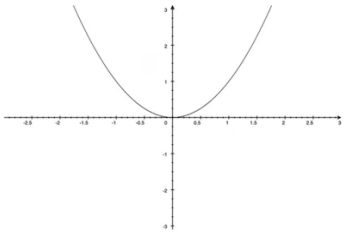
$z = 0$, xy(plane) $\quad$ $x = 0$, yz(plane) $\quad$ $y = 0$, xy(plane)

$0 = x^2 + y^2$

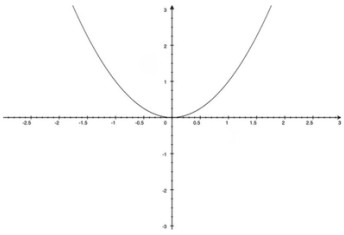
$x^2 = -y^2$

No solution in $\mathbb{R}$.

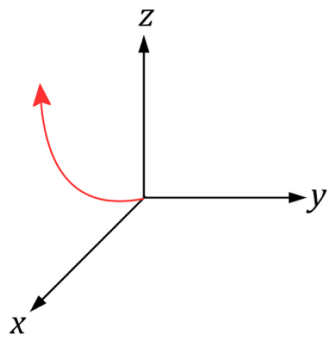
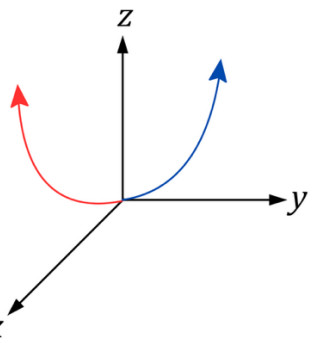
$z = x^2$

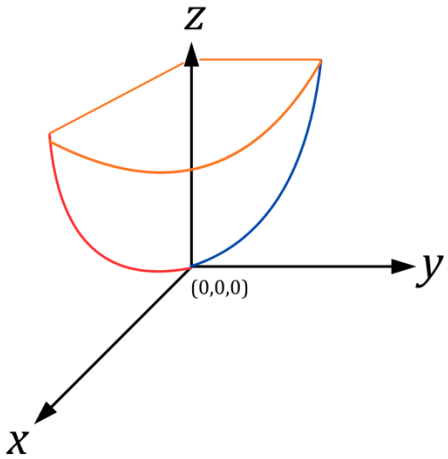


$z = x^2$



Now, to draw this in 3D, you would generally stick to octant one and draw a 3D system by tracing the graph in each plan and then connect the planes as follows.

xy-plane	xz-plane	yz-plane
A point at the origin.		



Next, you connect the curves. This works for pretty much any 3D graphing. It is usually called the traces.

Drawing by hand will require you to be methodic and make sure to mimic the professor's strategy in class as techniques change professor to professor.

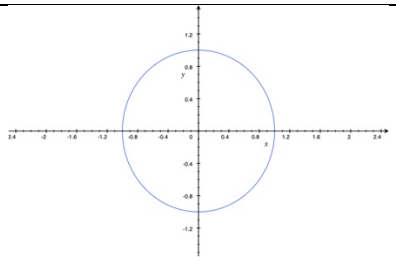
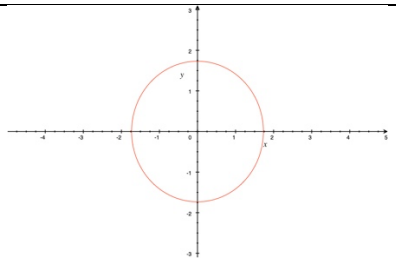
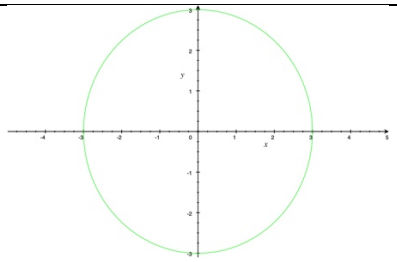
Now, we will look at how the gradient relates graphically to a 3D and a 2D object.

Let $z = x^2 + y^2$ which is a paraboloid positively oriented above the xy-plane.

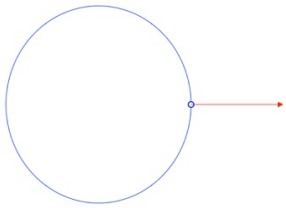
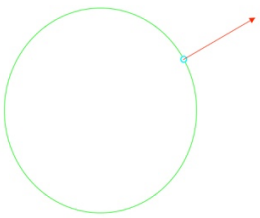
Then, the gradient is $\nabla z = \langle 2x, 2y \rangle$. We will now look at the contour graphs of z.

The gradient gives the direction of the fastest increase of the function and when you evaluate the gradient at a point on the contour graph, the vector at that point is orthogonal (perpendicular) to the curve.

To plot the contour graphs, let $z = 0,1,2, \ldots$ or whatever works easiest. In this case, $z > 0$.

$z = 1 \Rightarrow 1^2 = x^2 + y^2$	$z = 4 \Rightarrow 2^2 = x^2 + y^2$	$z = 3 \Rightarrow 3^2 = x^2 + y^2$
		

Now, we will plot the gradient vector field. You can see from the above images that these are slices in the xy-plane of the paraboloid. The gradient vector(s) will be perpendicular to each curve. Since we are dealing with circles, lets stick with the unit circle points at common angles, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ in quadrant one.

$P_1(1,0), \quad \nabla z(1,0) = \langle 2(1), 2(0) \rangle = \langle 2,0 \rangle$	$P_2(\sqrt{3}, 1), \quad \nabla z(1,0) = \langle 2\sqrt{3}, 2 \rangle$
	

What you see is the gradient of a 3D object when evaluated in 2D at points on the contour curve, is that the gradient vector field is always perpendicular to the contour curve.

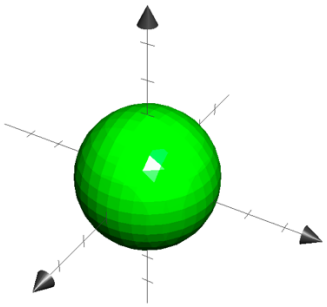
Similarly, in 4D the gradient vector field in 3D is perpendicular to the contour surfaces.

E.g. (for example) Let $w = x^2 + y^2 + z^2$. What does w represent graphically? This does not represent anything graphically as it is a fourth dimensional set of points which we as 3D objects that interpret everything one dimensionally are not capable of placing physical orientation to.

What you are thinking is the physical representation of the domain of the function is a sphere. And the gradient of w will be a vector field perpendicular to the sphere at all plots.

What you are thinking is the physical representation of the domain of the function is a sphere. And the gradient of w will be a vector field perpendicular to the sphere at all plots.

If you plot $\nabla w = \langle 2x, 2y, 2z \rangle$, you will see all vectors are orthogonal to the surface at all points just as in the previous examples.



Going forward, do not associate mathematics with graphs or physics representation as we are not in physics. You must learn to reference definitions, theorems, properties, axioms, ...

Partial Derivatives

First off, we need a good understanding of how derivatives work in standard and partial form.

Power Rule "Standard Differentiation"

$$\frac{d}{dx}x^n = \frac{d}{dx}(x)^n = nx^{n-1}\frac{d}{dx}x = nx^{n-1}\frac{dx}{dx} = nx^{n-1}(1) = nx^{n-1}$$

Power "Implicit Differentiation"

$$\frac{d}{dy}x^n = \frac{d}{dy}(x)^n = n(x)^{n-1}\frac{d}{dy}x = nx^{n-1}\frac{dx}{dy} = nx^{n-1}x'(y) = nx^{n-1}x'$$

$$\frac{d}{dx}y^n = \frac{d}{dx}(y)^n = n(y)^{n-1}\frac{d}{dx}y = ny^{n-1}\frac{dy}{dx} = ny^{n-1}y'(x) = ny^{n-1}y'$$

Product Rule "Implicit Time Derivatives"

$$\frac{d}{dt}(xy)^n = \frac{d}{dt}x^n y^n = y^n\frac{d}{dt}x^n + x^n\frac{d}{dt}y^n = y^n\left[nx^{n-1}\frac{dx}{dt}\right] + x^n\left[ny^{n-1}\frac{dy}{dt}\right]$$

NOTE Time Derivatives "Fly swat notation or dot notation"

$$\dot{x} \equiv x'(t) \equiv \frac{dx}{dt}\ \text{[velocity]}, \qquad \ddot{x} \equiv x''(t) \equiv \frac{d^2x}{dt^2}\ \text{[acceleration]}, \qquad \dddot{x} = \text{[jerk]}.$$

Now, partial derivatives. Not to be confused with $\frac{d}{dt}$ and $\frac{\partial}{\partial t}$. Standard d, derivative vs. partial ∂, (del) derivatives.

$$\frac{\partial}{\partial x}x^n = \frac{d}{dx}x^n = nx^{n-1}, \qquad \frac{\partial}{\partial y}x^n = x^n\frac{\partial}{\partial y}(1) = x^n\frac{d}{dy}(1) = x^n(0) = 0.$$

$$\frac{\partial}{\partial x}x^n y^n = y^n\frac{\partial}{\partial x}x^n = y^n\frac{d}{dx}x^n = y^n n x^{n-1}.$$

Students confuse implicit differentiation with partial derivatives because professor constantly make the mistake of saying ∂ (del), is 'd' and that the symbols are the same when they are not.

I.e.,

$$\frac{d}{dt}x^n y^n \neq \frac{\partial}{\partial x}x^n y^n.$$

[RP] You should now go solve chain rule partial derivative problems for practice from [1],[4].

Vector Calculus

Introduction to Notation

Before beginning line integrals from vector calculus, it is good to acknowledge that there are two approaches and many levels.

• **Approach 1**: Application of formulae.

• **Approach 2**: Deriving nth dimensional formulae.

At the several or multivariable level of line integral (anywhere between high school and junior year), you will simply need to apply formulae.

To begin this journey of methodic problem solving, we will start with application of formulae via theorem and definition. We will work from the Stewart Calculus [17] to derive an algorithmic approach for study and exam preparation in relation to solving line integrals and working with vector functions.

The approach: We will go up the ladder of knowledge and extract a step-by-step process of elimination to execute the solution.

Before we begin, let us understand some important notation. Remember: we are working from [17], so we will need to keep all notation synonymous with the text.

Symbols and Notations

Vector Function $\mathbf{F} =$	Gradient $\nabla f = \langle f_x, f_y, f_z \rangle$	Del Operator ∇
$\langle x(t), y(t), z(t) \rangle$	$\nabla f = \mathbf{F} \Leftrightarrow \mathbf{F}$ = conservative	$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$
$\langle P(x,y,z), Q(x,y,z), R(z,y,z) \rangle$	**NOTE** Conservative is a definition; not a concept.	**NOTE** This is not a vector
Same for 2D just kill R	Same for 2D just kill f_z	Symbol ∇ is nabla

For the unit vectors, this book uses non italicized bold lower case i, j, k—that is,

$$\mathbf{i} = \langle 1,0,0 \rangle, \qquad \mathbf{j} = \langle 0,1,0 \rangle, \qquad \mathbf{k} = \langle 0,0,1 \rangle.$$

Thus,

$$\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(z,y,z) \rangle = \langle P(x,y,z), 0,0 \rangle + \langle 0, Q(x,y,z), 0 \rangle + \langle 0,0, R(z,y,z) \rangle$$

$$= \langle P, 0,0 \rangle + \langle 0, Q, 0 \rangle + \langle 0,0, R \rangle = P\langle 1,0,0 \rangle + Q\langle 0,1,0 \rangle + R\langle 0,0,1 \rangle = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}.$$

Vector Fields

QUESTION [17] 16.1.9

Sketch the vector field **F** by drawing a diagram like Figure 5 or Figure 9.

$$\mathbf{F} = x\mathbf{k}.$$

Figure 5

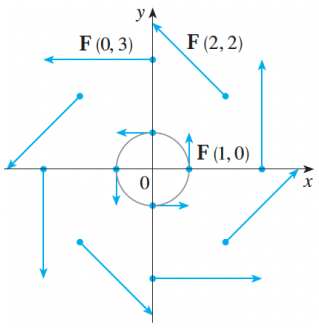
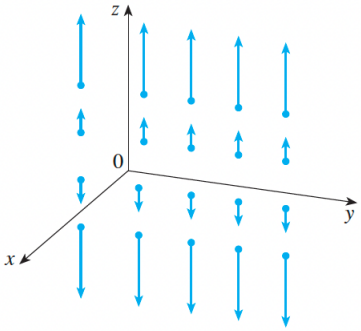
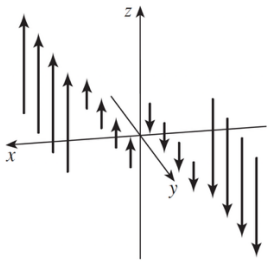


Figure 9



ANSWER



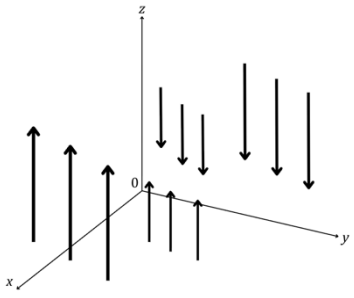
NOTE The questions figure 5 or 9. Thus, we don't have to label the coordinates for the 3D case.

Drawing this is a bit tedious by hand. You may have to on an exam if you are still living in the 1800s. The important part of this problem is identifying the vectors.

Let us now find some vectors. It is fairly straightforward—that is, you just evaluate coordinates and then plot.

$P_1(0,0,0)$	$P_2(1,1,1)$	$P_3(-1,-1,-1)$	$P_4(1,0,1)$
$\mathbf{F}_1 = \langle 0,0,0 \rangle$	$\mathbf{F}_2 = \langle 0,0,1 \rangle$	$\mathbf{F}_3 = \langle 0,0,-1 \rangle$	$\mathbf{F}_4 = \langle 0,0,1 \rangle$
$P_1(0,0,2)$	$P_2(0,0,2)$	$P_3(-2,-2,-2)$	$P_4(2,0,2)$
$\mathbf{F}_1 = \langle 0,0,2 \rangle$	$\mathbf{F}_2 = \langle 0,0,2 \rangle$	$\mathbf{F}_3 = \langle 0,0,-2 \rangle$	$\mathbf{F}_4 = \langle 0,0,2 \rangle$

It is pretty clear the pattern. We will now use a graphing application or Canva.com to plot these vectors. It may be a little tricky. On the negative side of the x—plane, the vectors are pointing in the negative direction and increasing in size as the go out to infinity. The opposite direction on the positive side with the same increase in length.



If you want to plot the coordinates

Step 1: Navigate to the chosen point. E.g., $P(1,1,1)$.

Step 2: Plot the vector start at that point. I.e., plot $\mathbf{F} = \langle 0,0,1 \rangle$ at the point $P(1,1,1)$.

Vector Calculus Crash Course

From the previous introductory lesson, we can see a standardized approached to problem solving. Going forward, we are going to simply solve the questions quickly as if they are exam questions.

There are two variations learning math-based subjects.

1) Applying the formulae.
2) Understanding the theorems, definitions, properties, rules, ...

At this level of mathematics, you are in the last stages of simple formulaic application. Then, you dive into a much more comprehensive approach to problem solving where you are required to understand all steps leading to the answer and then showing the answer is in fact correct—"The Full Circle."

For this section of the 'notations' volume, we are going to take a question from each sub-section of each section of each chapter and speed solve it. The point is to get familiar with how to quickly apply the theorems and definitions and equations(formulae) they produce.

Once we have a full understanding of how to apply the theorem, we will then look into the full circle method of problem solving.

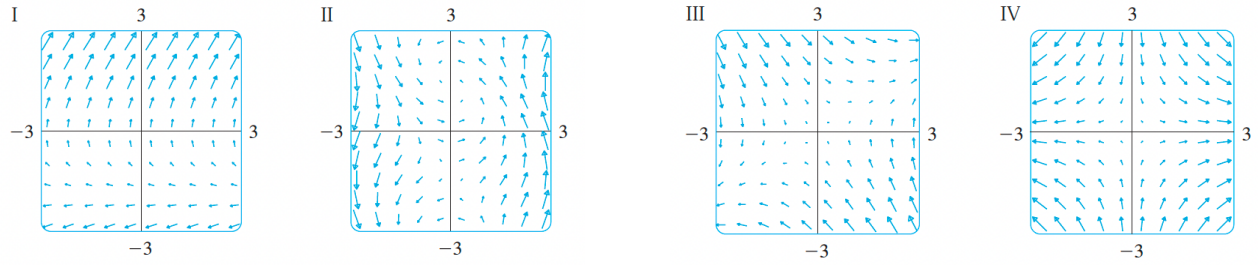
Rather than provide an in-depth explanation as in the previous question on plotting a vector field, we will simply solve the questions as if they are exam questions; quick but detailed.

This section is entirely based on chapter 16 from the Stewart Calculus textbook [17].

QUESTION [17] 16.1.13

Match the vector fields **F** with the plots labeled I–IV. Give reasons for your choices.

$$\mathbf{F}(x, y) = \langle y, y + 2 \rangle$$



ANSWER

I

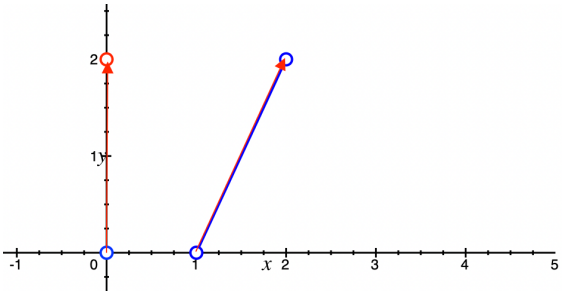
To match a vector function with its field,

1) Plot a point,
2) evaluate the function at the point,
3) plot the vector using the point as the starting point (where the tail begins).

For the points, choose simple numbers such as (0,0), (1,0), (1,1), … first.

(0,0)	(1,0)	(1,1)	(0,1)	(−1,0)	(−1, −1)
⟨0, 2⟩	⟨1, 2⟩				

If I go to the origin and plot vector ⟨0,2⟩ and look at the graph options, it would appear, right off the bat, that options II, III, and IV are nullified but since they never plotted any origin vectors, we cannot be sure. We plot more.



This is more than plenty to match the graph as we can immediately eliminate options II, III, and IV.

Thus, the answer is I.

We will now move to section 16.2. The concepts of plotting vectors is not as important. Continue at your own interest.

QUESTION [17] 16.2.1

Evaluate the line integral, where is the given curve.

$$\int_C y^3\,ds, \qquad C: x = t^3, \qquad y = t\,, \qquad 0 \le t \le 2.$$

ANSWER

$$\frac{1}{54}\left(145^{\frac{3}{2}} - 1\right)$$

$\mathcal{F}_1$ Formulae 50% of Credit on Exams

Recognize this is within the cartesian coordinate system of 2-dimensions. Thus,

$$\int_C f(x,y)\,ds = \int_a^b f\big(x(t), y(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

NOTE In order to identify the correct formula, you must examine the form of the integrand and differential. I.e. $\int_C$[integrand] [differential] $= \int_C y^3\,ds$.

So, how do we know this is the correct formula? Well, it is because it is in the same form with the same conditions. I.e.,

$$\int_C f(x,y)\,ds, \qquad C: t_1 \le t \le t_2 = \int_C y^3\,ds, \qquad C: 0 \le t \le 2.$$

We will call this the standard or basic form and start our process of elimination technique to algorithm building. Reference the end of this section for the elimination process list.

$\mathcal{F}_2$ Formulate 50% of Credit on Exams

EVERYTHING BELOW THIS LINE IS WRONG ↓

Thus,

$$\int_C y^3\,ds, \qquad C: x = t^3, \qquad y = t\,, \qquad 0 \le t \le 2.$$

$$\Rightarrow \int_C y^3\,ds = \int_0^2 f(t^3, t)\sqrt{(3t^2)^2 + (t)^2}\,dt = \int_0^2 (t)^3\sqrt{(3t^2)^2 + (t)^2}\,dt$$

$$= \int_0^2 t^3\sqrt{9t^4 + t^2}\,dt = \int_0^2 t^3\sqrt{9t^2\left(t^2 + \frac{1}{9}\right)}\,dt = \int_0^2 t^3\sqrt{9t^2}\sqrt{\left(t^2 + \frac{1}{9}\right)}\,dt$$

[PL] Students always forget to acknowledge the absolute value $\sqrt{t^2} \equiv |t|$. This is the kind of look over that takes an A student to a B student.

$$= 3\int_0^2 t^3|t|\sqrt{\left(t^2 + \frac{1}{9}\right)}\,dt, \qquad t \geq 0 \in [0,2]. \quad \therefore |t| = t.$$

$$\therefore \quad 3\int_0^2 t^3|t|\sqrt{\left(t^2 + \frac{1}{9}\right)}\,dt = 3\int_0^2 t^4\sqrt{\left(t^2 + \frac{1}{9}\right)}\,dt.$$

Now, I went a bit overboard. I did not need to do this. My mind is latched onto a more complicated version. Let us backtrack and apply "The Substitution Method."

Let $u = 9t^4 + t^2$. Then, $du = 36t^3 + 2t$, $u(2) = 9(2)^4 + 4$, $u(0) = 0$. Thus,

Never mind, this is not substitution. I was doing it correctly the first time.

So,

$$\int_0^2 t^3\sqrt{9t^4 + t^2}\,dt = \int_0^2 t^3\sqrt{t^2}\sqrt{(3t)^2 + 1}\,dt = \int_0^2 t^3|t|\sqrt{(3t)^2 + 1}\,dt$$

$$= \int_0^2 t^4\sqrt{(3t)^2 + 1}\,dt.$$

We now look into the process of elimination for standard integration.

1. Is it Integratable as is with formula?
2. The Substitution Method
3. Integration by Parts
4. Manipulation
5. Trig Identities
6. **Trigonometric Substitution**
7. Partial Fractions or Long Division
8. Other techniques, ...

This is clearly a trigonometric substitution. We now have to go back to the section for this situation and extract the formula.

From 7.3: $\sqrt{a^2 + x^2}$, $x = a\tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $1 + \tan^2\theta = \sec^2\theta$.

This tells us that $3t = \tan\theta$ and $a = 1$. Then $dt = \frac{1}{3}\sec^2\theta\, d\theta$ and $\theta = \arctan(3t)$.

$$\theta_1 = \arctan\big(3(2)\big) = \arctan(6)\,, \qquad \theta_2 = \arctan\big(3(0)\big) = 0$$

Now,

$$\int_0^2 t^4\sqrt{(3t)^2+1}\,dt = \int_0^{\arctan(6)} \left(\frac{1}{3}\tan\theta\right)^4 \sqrt{(\tan\theta)^2+1}\left(\frac{1}{3}\sec^2\theta\right)d\theta$$

$$= \frac{1}{3^4}\int_0^{\arctan(6)} \tan^4\theta\sqrt{\sec^2\theta}\left(\frac{1}{3}\sec^2\theta\right)d\theta = \frac{1}{3^5}\int_0^{\arctan(6)} \tan^4\theta\,|\sec\theta|\,(\sec^2\theta)d\theta$$

$$= \frac{1}{3^5}\int_0^{\arctan(6)} \tan^4\theta\sec^3\theta\,d\theta = \frac{1}{3^5}\int_0^{\arctan(6)} \frac{\sin^4\theta}{\cos^4\theta}\frac{1}{\cos^3\theta}d\theta$$

$$= \frac{1}{3^5}\int_0^{\arctan(6)} \frac{\sin^4\theta}{\cos^7\theta}d\theta$$

EVERTHING ABOVE THIS LINE IS WRONG ↑

I made a typo right at the beginning. I leave the errors so you can see that the math is not wrong, only I made type when I took the derivative of $y = t$. It should be $y' = 1$. I put $y = t$.

MISTAKE I did not take the derivative of y before I put it in.

$$\int_C y^3\,ds = \int_0^2 (t)^3\sqrt{(3t^2)^2+(t)^2}\,dt$$

[PL] Take note of the ridiculously silly mistake; if this were an exam question, I would have wasted a half hour doing nothing! You want to be meticulous and patient on exams to avoid such a silly mistake.

$$\therefore \int_0^2 t^3\sqrt{9t^4+1}\,dt.$$

Use the substitution method. Let $u = 9t^4 + 1$. Then,

$$du = 36t^3dt \Leftrightarrow \frac{1}{36}du = t^3dt, \qquad u(0) = 9(0)^4 + 1 = 1, \qquad u(2) = 9(2)^4 + 1$$

$$\Rightarrow \int_1^2 t^3\sqrt{9t^4+1}\,dt = \frac{1}{36}\int_1^{9(2)^4+1} u^{\frac{1}{2}}\,du = \frac{1}{36}\left[\frac{u^{\frac{1}{2}+\frac{2}{2}}}{\frac{1}{2}+\frac{2}{2}}\right]_1^{9(16)+1}$$

$$= \frac{1}{36}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{3^2\cdot 4^2+1} = \frac{1}{36}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{(12)^2+1} = \frac{1}{36}\left(\frac{2}{3}\right)\left[u^{\frac{3}{2}}\right]_1^{144+1} = \frac{1}{36}\left(\frac{2}{3}\right)\left[(145)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right]$$

$$= \frac{1}{18}\left(\frac{1}{3}\right)145^{\frac{3}{2}}-1 = \frac{1}{2(3^3)}145^{\frac{3}{2}}-1 = \frac{1}{54}145^{\frac{3}{2}}-1.$$

$\mathcal{F}_3$ Finalize 0% of Credit on Exams

Thus, by the basic line integral theorem,

$$\int_C y^3\,ds, \qquad C: x=t^3, \qquad y=t\,, \qquad 0\le t\le 2 = \frac{1}{54}145^{\frac{3}{2}}-1.$$

Now, for practice solve some questions from the 1-16 problem bank from chapter 16.2 in book [17]. Or, any line integral textbook you prefer.

$\mathcal{F}_1$ Formulae

$\mathcal{F}_2$ Formulate

$\mathcal{F}_3$ Finalize

Line Integral Process of Elimination "The Eye of The Integral"

• [1] Basic (or standard 2D/3D) Form

$$\int_C f(x,y,z)\,ds = \int_a^b f\big(x(t),y(t),z(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt.$$

END Road to Maxwell's Equations (will continue in the further editions)

Exact Equations in Differential Equations

QUESTION [21] 2.6.7 Determine whether the equation is exact. If it is exact, find the solution.

$$\left(\frac{y}{x}+6x\right)+(\ln x-2)y'=0, \qquad x>0.$$

ANSWER

$$y\ln x+3x^2-2y=c.$$

Before we begin to solve this question, we should acknowledge the condition, $x>0$. What does this mean? Why is it stated?

Because the logarithmic function $f(x)=\log x$ is defined to be only valid for values $x>0$, the condition is stated.

NOTE $\log x \equiv \log_e x \equiv \ln x \equiv \frac{\ln x}{\ln e}$. Change of base $\log_b a = \frac{\ln a}{\ln b}$. $\log x \equiv \ln x$. Base 10 is a TI-84 thing. $\log x$ and $\ln x$ are the same. $\log x$ is base e. For $\log x$ or $\ln x$, the logarithmic function is defined to be $x>0$ for all x. Thus, the acknowledgment of the restriction.

For differential equations, it is important to first cite the theorem(s)/definition(s) prior to attempting to solve.

EXACT EQUATION Theorem 2.6.1

$$M(x,y)+N(x,y)y'=0$$

Exact iff $M_y = N_y$ iff $\psi_x = M,\ \psi_y = N$

Also, ψ_{xy}, ψ_{yx} are continuous.

The verbatim theorem above is what we will work from. It is good to recognize that this is the same exact problem as solving conservative vector function in multivariable calculus.

Step 1 for Differential Equations: Identify the ODE,

$$\left(\frac{y}{x} + 6x\right) + (\ln x - 2)y' = 0, \qquad x > 0.$$

The above equation with dependent variable, y and independent variable x says that the ODE is first-order-linear-nonhomogeneous. We can put it into exact form from different books as well. We are in [21].

$\mathcal{F}_1$ Identify the ODE "first-order-linear-nonhomogeneous" with potential exactness.

Next, identify the method needed to solve the ODE. In this case, we told what to do. However, that may not be the case on exams. You should take account for formation to be prepped for exam alteration of structure.

$$M(x,y) = \frac{y}{x} + 6x, \qquad N(x,y) = (\ln x - 2).$$

$\mathcal{F}_2$ Formulation of the theorem(s)/definition(s)/equation(s)

The equation is exact iff $M_y = N_y$.

$$M_y = \frac{\partial}{\partial y}\left(\frac{y}{x} + 6x\right) = \frac{\partial}{\partial y}\frac{y}{x} + \frac{\partial}{\partial y}6x = \frac{1}{x}\frac{\partial}{\partial y}y + 6x\frac{\partial}{\partial y}(1) = \frac{1}{x}\frac{d}{dy}y + 6x\frac{d}{dx}(1)$$

$$= \frac{1}{x}\frac{dy}{dy} + 6x(0) = \frac{1}{x}(1) + (0) = \frac{1}{x}.$$

$$N_y = \frac{\partial}{\partial x}(\ln x - 2) = \frac{\partial}{\partial x}\ln x - \frac{\partial}{\partial x}2 = \frac{d}{dx}\ln x - (0) = \frac{1}{x}\frac{d}{dx}x = \frac{1}{x}\frac{dx}{dx} = \frac{1}{x}.$$

Thus,

$$M_y = \frac{1}{x} = \frac{1}{x} = N_x.$$

We can now see by the theorem that the equation is indeed exact.

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So, how do we solve it? Well, this is the same as a conservative function in many variable calculus techniques. But, we are not in calculus-MV, so we must stick with the book!!!

Let $\psi_x = M$ and $\psi_y = N$

$$\psi_x = M(x,y) = \frac{y}{x} + 6x, \qquad \psi_y = N(x,y) = \ln x - 2.$$

Now, we have a relation of antiderivative nature. Let us take the antiderivatives.

$$\psi_x = \frac{y}{x} + 6x, \qquad \psi_y = \ln x - 2 \quad \Rightarrow \quad \psi = \int \frac{y}{x} + 6x \, \partial x, \qquad \psi = \int \ln x - 2 \; \partial y$$

$$\Rightarrow \quad \psi = \int \frac{y}{x} + 6x \, dx, \qquad \psi = \int \ln x - 2 \; dy$$

$$\Rightarrow \quad \psi = \int \frac{y}{x} dx + \int 6x \, dx, \qquad \psi = \int \ln x \, dy - \int 2 \; dy$$

$$\Rightarrow \quad \psi = y \int \frac{1}{x} dx + 6 \int x \, dx, \qquad \psi = \ln x \int dy - 2 \int dy$$

$$\Rightarrow \quad \psi = y \ln|x| + 6\left[\frac{1}{2}x^2\right] + g_1(y), \qquad \psi = \ln x \, [y] - 2[y] + g_2(x)$$

We now have two equation that are the same but with mix and match variable.

$$\psi = y \ln|x| + 3x^2 + g_1(y), \qquad \psi = y \ln x - 2y + g_2(x)$$

$$\Rightarrow \quad \psi = y \ln|x| + 3x^2 + g_1(y), \qquad \psi = y \ln x + (-2y) + g_2(x)$$

$$\Rightarrow \quad \psi = y \ln|x| + 3x^2 + g_1(y), \qquad \psi = y \ln x + (-2y) + g_2(x)$$

$$h(x,y) = y \ln x.$$

Thus, you can omit the absolute value since the original statement states $x > 0$, we know that it is positive so we can drop the absolute value. (Don't' forget the scalar constant.)

$$\therefore \psi = y \ln x + 3x^2 - 2y + C.$$

NOTE I have written the answer as $\psi = y \ln x + 3x^2 - 2y + C$, however, the books answer is $y \ln x + 3x^2 - 2y = c$.

Do you know why?

$\mathcal{F}_3$ Finalize the answer stated correctly.

In a 3D calculus situation, it is common to add the constant and equate to a function.

$$y\ln x + 3x^2 - 2y = c \quad vs \quad \psi = y\ln x + 3x^2 - 2y + C$$

The former is the same relation—that is, one is state as a function that equals zero whereas the other is stated as a function which equals zero.

$$\psi = y\ln x + 3x^2 - 2y + C = 0, \qquad y\ln x + 3x^2 - 2y - c = 0.$$

NOTE A constant with a constant is a constant. $-c$ is a variation of C. $\psi = 0$ in this case. Depending on the course you are in, you will choose to label. We are master of labeling.

TEST RESPONSE

QUESTION Solve the ODE

$$\left(\frac{y}{x} + 6x\right) + (\ln x - 2)y' = 0, \qquad x > 0.$$

(Step 1) By exact equation $M + Ny' = 0$ if $M_y = N_x$, then it is exact.

(Step 2)

$$\psi_x = M(x,y) = \frac{y}{x} + 6x, \qquad \psi_y = N(x,y) = \ln x - 2.$$

$$\Rightarrow \psi = \int \frac{y}{x} + 6x\, dx, \qquad \psi = \int \ln x - 2\ dy$$

$$\Rightarrow \psi = \int \frac{y}{x} dx + \int 6x\, dx, \qquad \psi = \int \ln x\, dy - \int 2\ dy$$

$$\Rightarrow \psi = y\int \frac{1}{x} dx + 6\int x\, dx, \qquad \psi = \ln x \int dy - 2\int dy$$

$$\Rightarrow \psi = y\ln|x| + 6\left[\frac{1}{2}x^2\right] + g_1(y), \qquad \psi = \ln x\,[y] - 2[y] + g_2(x)$$

$$\Rightarrow \psi = y\ln|x| + 3x^2 + g_1(y), \qquad \psi = y\ln x - 2y + g_2(x)$$

$$\Rightarrow \psi = y\ln|x| + 3x^2 + g_1(y), \qquad \psi = y\ln x + (-2y) + g_2(x)$$

$$\Rightarrow \psi = y\ln|x| + 3x^2 + g_1(y), \qquad \psi = y\ln x + (-2y) + g_2(x).$$

$\mathcal{F}_3$ Finalize and state the answer with citation.

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By "The Exact Theorem"

$$y \ln x + 3x^2 - 2y = c.$$

[FC] How do we know this is right? The book matches but how do we know?

It is now your turn to show the answer satisfies the question.

Show that $\{e^x, e^{-x}, -e^x\}$ is a linearly dependent set.

$$\text{span}\{e^x, e^{-x}, -e^x\} = c_1e^x + c_2e^{-x} + c_3(-e^x) = c_1e^x + c_2e^{-x} + c_4e^x$$

$$= (c_1 + c_4)e^x + c_2e^{-x} = c_5e^x + c_2e^{-x} \quad \Rightarrow \quad \{e^x, e^{-x}, -e^x\} \sim \{e^x, e^{-x}\}.$$

Thus, the set $\{e^x, e^{-x}, -e^x\}$ is a linearly dependent set.

In general—that is, reference your textbook for the proper/formal definition/theorem(s).

For the above, if the solution to a differential equation is $y = c_1e^x + c_2e^{-x} + c_3(-e^x)$,

Then,

$$y = c_1e^x + c_2e^{-x} + c_3(-e^x) = c_1e^x + c_2e^{-x} + c_4e^x$$

$$= (c_1 + c_4)e^x + c_2e^{-x} = c_5e^x + c_2e^{-x} = \begin{bmatrix} e^x & e^{-x} \end{bmatrix} \begin{bmatrix} c_2 \\ c_5 \end{bmatrix} = \begin{bmatrix} e^x & e^{-x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Also note: $\text{span}\{e^x, e^{-x}, -e^x\} = c_1e^x + c_2e^{-x} + c_3(-e^x)$.

If you are in a differential equations course with linear algebra, the techniques and definitions are going to change from your standard linear algebra courses.

NOTE At this part of the book, since I am not referencing a textbook, the notation would be classified as slang use. The point is to get you familiar with referencing multiple sources.

NOTE If you are prompted to see if something is within the span of a set, you simple set the span of the set equal to the object. Then, check if there is a unique solution.

E.g., is e^{2x} in the span $\{e^x, e^{-x}, -e^x\}$?

$$\begin{bmatrix} e^x & e^{-x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} e^{2x} \end{bmatrix} \quad \Leftrightarrow \quad c_5e^x + c_2e^{-x} = e^{2x}$$

$$\Rightarrow \begin{bmatrix} e^x & e^{-x} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = (1 \times 2)(2 \times 2)(2 \times 1) = (1 \times 1) = e^{2x}$$

$$\Rightarrow \begin{bmatrix} e^x & e^{-x} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} e^{2x} & * \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

I.e. $e^{2x} \notin \text{span}\{e^x, e^{-x}, -e^x\}$. There are no constants that could multiply $\{e^x, e^{-x}, -e^x\}$ any of these to make them equal. How come $c_1 = e^x$ making $c_1 e^x = e^x e^x = e^{x+x} = e^{2x}$?—because that is not a constant, it is a varying function. The question does not make sense to begin with because it is like doing calculations between $\mathbb{R}^3$ and $\mathbb{R}^5$. They are not compatible.

QUESTION [19] R.1.19

Use $U = \{0,1,2,3,4,5,6,7,8,9\}, A = \{1,3,4,5,9\}, B = \{2,4,6,7,8\}, C = \{2,4,6,7,8\}$ to find $\bar{A} \cup \bar{B}$.

ANSWER

$$\{0,1,2,3,4,5,6,7,8,9\}.$$

NOTE Before you begin to learn mathematics, you must understand you cannot leave the textbook. You must also pay attention to anything in a box (or highlighted)—that is, definitions, theorems and equations.

$\mathcal{F}_1$ Formulae: Find the definitions, theorems, properties, rules, laws, axioms, ... needed to solve the problem.

Union $\cup$

When two sets are in union, all elements are included. I.e.,

$$A = \{x, y, a_1, b_1, a_2, b_2 \ldots\} \cup \{1,2, b_1, b_1, b_2, \ldots\} = B \Rightarrow A \cup B = \{x, y, 1,2, a_1, b_1, a_2, b_2, \ldots\}.$$

Complement $\bar{A}$

If A is a set, the complement of A, denoted $\bar{A}$ is the set consisting of all the elements in the universal set that are not in A.

Let $A = \{1,2,3\}$, $U = \{1,2,3,4,5,6\}$. Then, $\bar{A} = U \setminus A = \{1,2,3,4,5,6\} \setminus \{1,2,3\} = \{4,5,6\}$.

Set Minus (\setminus) "\"

$\mathcal{F}_2$ Now, use the previous information to solve the problem or write the solution.

Find $\bar{A} \cup \bar{B}$.

$$\bar{A} = U \setminus A = \{0,1,2,3,4,5,6,7,8,9\} \setminus \{1,3,4,5,9\} = \{0,2,6,7,8\},$$

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$$\bar{B} = U \setminus A = \{0,1,2,3,4,5,6,7,8,9\} \setminus \{2,4,6,7,8\} = \{0,1,3,5,9\}$$

$$\Rightarrow \quad \bar{A} \cup \bar{B} = \{0,2,6,7,8\} \cup \{0,1,3,5,9\} = \{0,1,2,3,4,5,6,7,8,9\}.$$

$\mathcal{F}_3$ State the answer with citation.

Therefore, according to compliment and union definition, $\bar{A} \cup \bar{B} = \{0,1,2,3,4,5,6,7,8,9\}$.

10.PE.47 [7] Power Series – Sequences and Series in Calculus

Radius and Interval of Convergence

QUESTION

(a) Find the series' radius and interval of convergence. Then identify the values of x for which the series converges (b) absolutely and (c) conditionally.

$$\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$$

ANSWER

$$R = \sqrt{3}, \qquad I = \left(-\sqrt{3}, \sqrt{3}\right)$$

[**Step 1: Process of Elimination**] For the most part, the ratio or root test are the tests to be used for interval of converge. Since this as a power of $2n-1$, the root test may or may not be the best route. When doing intervals of convergence, a good fail safe, is to simple use the ratio test.

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} < 1, & Abs\text{Converges} \\ > 1, & \text{Diverges} \\ = 1, & \text{Inconclusive} \end{cases}.$$

The above is worth 50% of the credit (unless the test doesn't work).

[**Step 2: Execute the Formula**]

$$a_n = \frac{(n+1)x^{2n-1}}{3^n} \Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\left((n+1)+1\right)x^{2(n+1)-1}}{3^{(n+1)}}}{\frac{(n+1)x^{2n-1}}{3^n}} \right|$$

$$= \lim_{n\to\infty}\left|\frac{\big((n+1)+1\big)x^{2(n+1)-1}}{3^{(n+1)}} \cdot \frac{3^n}{(n+1)x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+2-1}}{3^n\cdot 3} \cdot \frac{3^n}{(n+1)x^{2n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1}}{\cancel{3^n}\cdot 3} \cdot \frac{\cancel{3^n}}{(n+1)x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1}}{3} \cdot \frac{1}{(n+1)x^{2n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1}}{3} \cdot \frac{1}{(n+1)x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1}}{3(n+1)x^{2n-1}}\right|$$

[PL] At this point, people often forget to keep the absolute value on the variable.

$$= \lim_{n\to\infty}\left|\frac{(n+2)x^{(2n+1)-(2n-1)}}{3(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{2n+1-2n+1}}{3(n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x^2}{3(n+1)}\right| = \left|\frac{x^2}{3}\right|\lim_{n\to\infty}\frac{(n+2)}{(n+1)} = \left|\frac{x^2}{3}\right|\lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\frac{n}{n}$$

$$= \left|\frac{x^2}{3}\right|\frac{\left(\lim_{n\to\infty}1+\lim_{n\to\infty}1\frac{2}{n}\right)}{\left(\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n}\right)} = \left|\frac{x^2}{3}\right|\frac{(1+0)}{(1+0)} = \left|\frac{x^2}{3}\right| < 1.$$

In order for this to be convergent, $\left|\frac{x^2}{3}\right| < 1$. Solve the inequality.

$$\left|\frac{x^2}{3}\right| < 1 \;\Rightarrow\; |x^2| < 3 \;\Rightarrow\; x < \sqrt{3} \text{ or } x > -\sqrt{3}$$

$$\Rightarrow\; x < \sqrt{3}, \qquad x > \sqrt{-3}.$$

[PL] At this point, students always forget to test the endpoints. The answer does not include the endpoints. It was chosen to specifically set you up for failure. Why? Because, when the exam comes, you will need to test the endpoints and one, both or neither will be included. If you don't show this, you won't get full credit.

Endpoint $x = -\sqrt{3}$

$$\sum_{n=0}^{\infty}\frac{(n+1)x^{2n-1}}{3^n} \;\Rightarrow\; \sum_{n=0}^{\infty}\frac{(n+1)\big(-\sqrt{3}\big)^{2n-1}}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^{2n-1}(n+1)(3)^{n-1/2}}{3^n}$$

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$$= \sum_{n=0}^{\infty} \frac{(-1)^{2n-1}(n+1)3^n}{\sqrt{3}3^n} = \sum_{n=0}^{\infty} \frac{(-1)^{2n-1}(n+1)}{\sqrt{3}} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^{2n-1}(n+1)$$

We need to show this is divergent by using another test.

By the nth term for divergence, we see that the limit $(-1)^{2n-1}(n+1)$ does not exist. Thus, the series is divergent, and the endpoint is not included.

Endpoint $x = \sqrt{3}$

$$\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n} \Rightarrow \sum_{n=0}^{\infty} \frac{(n+1)\left(\sqrt{3}\right)^{2n-1}}{3^n} = \sum_{n=0}^{\infty} \frac{(n+1)(3)^{n-1/2}}{3^n}$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)}{\sqrt{3}} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} (n+1)$$

This is also divergent because the limit to infinity of $(n+1)$ is infinity or does not exist. Thus, by the nth term for divergence test, the series diverges and it is not included in the endpoint.

[**Step 3- Finalize**]

For part (a), we can clearly see that the radius is $R = \sqrt{3}$ since it is centered at the origin. And the interval is an open interval because the endpoints are divergent. Thus, $I = \left(-\sqrt{3}, \sqrt{3}\right)$.

For part (b), from the theorem of ratio test, we know that it is absolutely convergent for all values less than 1. Thus, it is absolutely convergent on the interval of convergence.

A series is conditionally convergent, when the series is convergent and

---

Conditional Convergence

A convergent series that is **not absolutely convergent** is conditionally convergent.

Since the series is convergent only on the interval found, and by definition of ratio test, it is also absolutely convergent on that interval, meaning it is not conditionally convergent for any x.

[R.P.] For those in the research position, you can now ponder and review to find any issue or error. We will reconnect to finalize in the next session.

Line Integral: Conservative Function 3D

QUESTION 16.5.17 [17] Determine whether or not the vector field is conservative. If it is conservative, find a function f such the $\mathbf{F} = \nabla f$.

$$\mathbf{F}(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$$

ANSWER

$$f(x, y, z) = xe^{yz} + K$$

NOTE: This question is Stewart Calculus—that is, using $\hat{\imath} = (1,0,0)$ or $\hat{\imath} = \langle 1,0,0\rangle$ (instead of $\boldsymbol{i} = \langle 1,0,0\rangle$ *would make the solution* incorrect as it is different definitions even though the answer is the same.

$\mathcal{F}_1$. Let us find the correct definition, formula, ...

How do we know what to look for? We read the question and seek the instructions via keywords. In this case, the words we want are **conservative** and **antiderivative**—i.e., $\nabla f \to f$.

[PL] Just because **F** can be equal to ∇f (the gradient), doesn't mean **F** is always the gradient. It has to be conservative to be the same. Meaning- ∇f can always be **F** but **F** is not always ∇f.

THEOREM 3 (Rephrased) **F** is conservative if the following is true.

$$\text{curl}(\nabla f) = \mathbf{0} \quad \overset{\uparrow}{\Rightarrow} \quad \text{curl}(\mathbf{F}) = \mathbf{0}.$$

NOTE What this means is, "If **F** is conversative, the antiderivative exists."

$\mathcal{F}_2$. Now formulate the formula.

The Curl (∇ by itself is called the del operator)

$$\text{curl}(\nabla f) = \nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}.$$

[PL] **NOTE** This is not the determinate of a 3×3 matrix as that would be scalar. This is a vector.

$$\mathbf{F}(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k} = \langle e^{yz}, xze^{yz}, xye^{yz}\rangle = \langle f_x, f_y, f_z\rangle$$

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[EXAMS] Before I brute computation this, I would like to examine shortcuts as this is tedious and often there are shortcuts when tedious question come about on exams.

A thought is that e^{yz} appears in all terms. Can we factor to simplify? Or, is there a trick of sorts from the reading?...

[PT] The following theorem may prevent you from unnecessary tedious calculation

---

TH4

> **4 Theorem** If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

The theorem says, if you have a discontinuous partial, it ain't be conservative! In our case, they are all continuous, $\langle e^{yz}, xze^{yz}, xye^{yz}\rangle$. But, we could avoid all the nasty work on an exam, by recognizing Theorem 4—that is, if one component's partial is discontinuous, are you have to do is make that statement by TH4 and get full credit. About 2-minutes of work vs. 20-minutes of pointless work. Why so many students destroy their education. E.g.,

---

Is $\mathbf{F} = \langle x, \ln y\,, e^x\rangle$ conservative?

Most students would do the curl, when all you have to do, is say, "By Theorem 4, $f_y = \frac{1}{y}$ is discontinuous at $y = 0$. Thus, $\mathbf{F}$ is not conservative. [FULL CREDIT 100%] Vs what is coming next.

NOTE The Del Operator is the set of partial derivative operators, $\langle \partial_x, \partial_y, \partial_z\rangle$

$$\operatorname{curl}(\nabla f) = \nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{yz} & xze^{yz} & xye^{yz} \end{vmatrix}$$

This is the same "arithmetic" as the "cross-product," but it is not a scalar. Thus, not a cross-product.

$$= +\left[\frac{\partial}{\partial y}(xye^{yz}) - \frac{\partial}{\partial z}(xze^{yz})\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(xye^{yz}) - \frac{\partial}{\partial z}(e^{yz})\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(xze^{yz}) - \frac{\partial}{\partial y}(e^{yz})\right]\mathbf{k}$$

$$= \left[\frac{\partial}{\partial y} xye^{yz} - \frac{\partial}{\partial z} xze^{yz}\right]\mathbf{i} - \left[\frac{\partial}{\partial x} xye^{yz} - \frac{\partial}{\partial z} e^{yz}\right]\mathbf{j} + \left[\frac{\partial}{\partial x} xze^{yz} - \frac{\partial}{\partial y} e^{yz}\right]\mathbf{k}$$

$$= \left[x\frac{\partial}{\partial y} ye^{yz} - x\frac{\partial}{\partial z} ze^{yz}\right]\mathbf{i} - \left[y\frac{\partial}{\partial x} xe^{yz} - \frac{\partial}{\partial z} e^{yz}\right]\mathbf{j} + \left[z\frac{\partial}{\partial x} xe^{yz} - \frac{\partial}{\partial y} e^{yz}\right]\mathbf{k}$$

$$= \left[x\frac{\partial}{\partial y} ye^{yz} - x\frac{\partial}{\partial z} ze^{yz}\right]\mathbf{i} - \left[ye^{yz}\frac{\partial}{\partial x} x - \frac{\partial}{\partial z} e^{yz}\right]\mathbf{j} + \left[ze^{yz}\frac{\partial}{\partial x} x - \frac{\partial}{\partial y} e^{yz}\right]\mathbf{k}$$

There is a lot of arithmetic that we could inspect to reduce time (maybe). Before I begin the product rule, I will see if I can simplify elsewhere. I don't see anything...

$$= \left[x\frac{\partial}{\partial y} ye^{yz} - x\frac{\partial}{\partial z} ze^{yz}\right]\mathbf{i} - \left[ye^{yz}(1) - \frac{\partial}{\partial z} e^{yz}\right]\mathbf{j} + \left[ze^{yz}(1) - \frac{\partial}{\partial y} e^{yz}\right]\mathbf{k}$$

$$= \left[x\frac{\partial}{\partial y} ye^{yz} - x\frac{\partial}{\partial z} ze^{yz}\right]\mathbf{i} - [ye^{yz} - ye^{yz}]\mathbf{j} + [ze^{yz} - ze^{yz}]\mathbf{k}$$

$$= \left[x\frac{\partial}{\partial y} ye^{yz} - x\frac{\partial}{\partial z} ze^{yz}\right]\mathbf{i} - (0)\mathbf{j} + (0)\mathbf{k}$$

Good chance this is zero:

$$x\frac{\partial}{\partial y} ye^{yz} - x\frac{\partial}{\partial z} ze^{yz} = x\left[\frac{\partial}{\partial y} ye^{yz} - \frac{\partial}{\partial z} ze^{yz}\right]$$

$$= x\left[y\frac{\partial}{\partial y} e^{yz} + e^{yz}\frac{\partial}{\partial y} y - \left(z\frac{\partial}{\partial z} e^{yz} + e^{yz}\frac{\partial}{\partial z} z\right)\right]$$

$$= x\left[yze^{yz} + e^{yz}(1) - \left(yze^{yz} + e^{yz}(1)\right)\right]$$

$$= x[yze^{yz} + e^{yz} - yze^{yz} - e^{yz}] = 0.$$

$\boldsymbol{\mathcal{F}_3}$ (part 1)

Thus,

$$\text{curl}(\mathbf{F}) = \mathbf{0}.$$

Thus, it is conservative.

Thus, we can find its antiderivative.

We now have to find the antiderivatives. (This is similar to solving a partial differential equation if not the same).

[RP] For you guys, you should now use the Stewart Calculus Textbook to see if you can find the antiderivative. I will continue this, in the next video.

TEST Response

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This shows the bare minimum of writing for 100% credit on exams.

Inverse Functions and Sketching [PRECALCULUS]

QUESTION Ch.2 Test #15

(a) If $f(x) = \sqrt{3 - x}$, find the inverse function f^{-1}.
(b) Sketch the graphs of f and f^{-1} on the same coordinate axes.

ANSWER

(a) $f^{-1}(x) = 3 - x^2, \qquad x \geq 0$ **(b)**

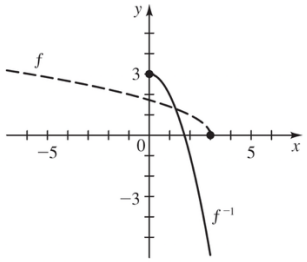


Image from...

SOLUTION

$\mathcal{F}_1$ [Formulae] Gather all **definitions**, theorems, equations, ...

DEFINITION Inverse Function

If f is one-to-one with domain A and range B, then f^{-1} has domain B and range A.

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

NOTE For f, $x \in [a, b]$, $y \in [c, d]$ and for f^{-1}, $x \in [c, d]$, $y \in [a, b]$

$\mathcal{F}_2$ [Formulate] Assemble the previous information into a puzzle and find the missing piece.

Now, we can solve the problem. For this solution, it is as simple as flip-flopping y and x and then solve for y.

[PL] We must first show the function is one-to-one in order to use the definition. However, the question implies that it is one-to-one as it states, "find the inverse." Since it states that, we don't need to show it is one-to-one. However, on an exam (or homework), the question may be posed as, "If the function is one-to-one, find the inverse." In this case, we would need to show that the function is one-to-one.

$$f(x) = \sqrt{3 - x} \quad \Rightarrow \quad y = \sqrt{3 - x} \quad \Rightarrow \quad x = \sqrt{3 - f^{-1}} \quad \Rightarrow \quad x = \sqrt{3 - y}$$

$$\Rightarrow \quad (x)^2 = \left(\sqrt{3 - y}\right)^2 \quad \Rightarrow \quad x^2 = 3 - y$$

NOTE $\sqrt{x^2} \equiv |x|,\ \left(\sqrt{x}\right)^2 = x.$

$$\Rightarrow \quad x^2 - 3 = 3 - 3 - y \quad \Rightarrow \quad x^2 - 3 = 0 - y \quad \Rightarrow \quad -y = x^2 - 3$$

$$\Rightarrow \quad (-1)[-y = x^2 - 3] \quad \Rightarrow \quad y = (-1)x^2 + 3$$

$$\Rightarrow \quad y = -x^2 + 3 = 3 - x^2 \quad \Rightarrow \quad f^{-1} = 3 - x^2.$$

Before going any further, address the domain and range of f.

For

$$\sqrt{g(x)}, \qquad D \coloneqq g \geq 0$$

Some other domain examples.

$\sqrt{u}, \quad D \coloneqq u \geq 0$	$\frac{1}{\sqrt{u}}, \quad D \coloneqq u > 0$	$\ln u\,, \quad u > 0$

Thus, the domain of $f(x) = \sqrt{3 - x}$

$$\Rightarrow \quad 3 - x \geq 0 \quad \Rightarrow \quad -x \geq -3 \quad \Rightarrow \quad x \leq 3.$$

$$A = (-\infty, 3], \qquad B = [0, \infty) \text{ (range)}.$$

Now, for f^{-1}, the range and domain flip.

Thus, $f^{-1} = 3 - x^2$ has a domain of $[0, \infty)$ or $x \geq 0$.

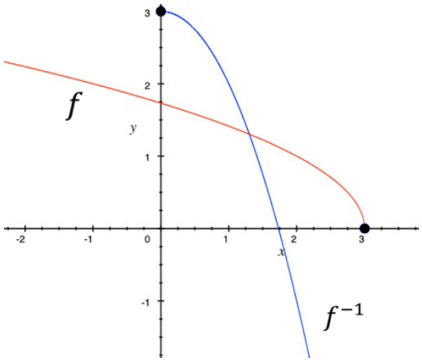
$\mathcal{F}_3$ [Finalize] Clearly state the answer with appropriate justification.

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For part **(a)** $f^{-1} = 3 - x^2,\ x \geq 0$.

NOTE Be careful not to graph the entire parabola as only half of it is on the graph via the domain.

For part **(b)**, the graph is show.



EXAM

[1]

Since f is one-to-one, by definition the domain and range of f is the range and domain of f^{-1}.

[2]

Thus,

$$f(x) = \sqrt{3-x} \quad \Rightarrow \quad y = \sqrt{3-x} \quad \Rightarrow \quad x = \sqrt{3-y}$$

$$\Rightarrow \quad (x)^2 = \left(\sqrt{3-y}\right)^2 \quad \Rightarrow \quad x^2 = 3 - y.$$

$$\therefore \quad f^{-1} = 3 - x^2.$$

Also, the domain of $f(x) = \sqrt{3-x}$

$$\Rightarrow \quad 3 - x \geq 0 \quad \Rightarrow \quad -x \geq -3 \quad \Rightarrow \quad x \leq 3.$$

$$A = (-\infty, 3], \qquad B = [0, \infty) \text{ (range)}.$$

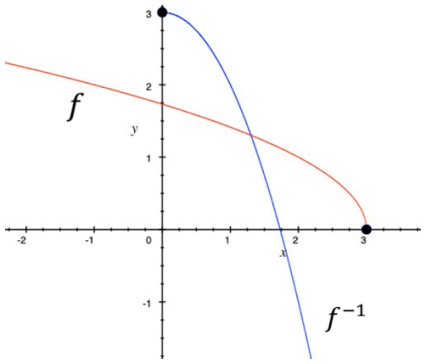
Now, for f^{-1}, the range and domain flip.

Thus, $f^{-1} = 3 - x^2$ has a domain of $[0, \infty)$ or $x \geq 0$.

[3]

Therefore, by definition,

For part **(a)** $f^{-1} = 3 - x^2, \ x \geq 0$.



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Introduction to Problem Solving Techniques: College Precalculus

Before we begin, we will locate a textbook and take a look at it. It should be known that Algebra with Trigonometry or College Algebra and Trigonometry are the same exact books as a standard Precalculus textbook; only, they may—on occasion—have an introduction to limits in calculus in the latter chapters. *This depends on the textbook/author*.

Precalculus is essentially just a more elaborate version condensed into one semester of both algebra and trigonometry. Meaning: it is a review course but at a more complicated level.

[28] Stewart, J., Redlin, L., & Watson, S. (2016). Precalculus: Mathematics for calculus (7th ed.). Cengage Learning.

https://opac.atmaluhur.ac.id/uploaded_files/temporary/DigitalCollection/YTAxYTBiOGM1MzE1ZmQ1ODk0ODA3ZGFkY2U4MDM0YWYwOWRkMmFlNA==.pdf

For this lesson, we are going to do a rather basic question that comes from either Algebra, Precalculus or Integration Calculus: Sequences and Series

This is a preliminary lesson to show students how to approach reading and solving a question from the precalculus textbook, [28].

QUESTION [28] 12.3.79 Express the repeating decimal as a fraction.

$$0.030303\ldots$$

ANSWER [Page A77]

$$\frac{1}{33}$$

SOLUTION

Before we began, we looked at sections 12.1, 12.2, and 12.3 to understand what we need to know. From skimming the sections, what I found was not a clear description of how to solve this problem, rather I found the reversal of it—meaning, we have to do a sort of reverse engineering. [COR] After review, the same solution was in the reading of 12.3.

NOTE It should also be noted that at the beginning of the textbook, the author explicitly says, "Work Backward" in the "Prologue: Principles of Problem Solving" section.

$\mathcal{F}_1$ [Formulae] For this question, we are in the Geometric Series section. So, we probably need the Geometric Series. Also note, $0.030303\ldots \equiv 0.\overline{03}$.

GEOMETRIC SERIES

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \quad \Leftrightarrow \quad |r| < 1.$$

If $|r| \geq 1$, the series is said to diverge.

For future reference (this may not apply now) in calculus and higher, the summation can be rewritten as follows,

$$\sum_{k=1}^{\infty} ar^{k-1} = \sum_{k=1-1}^{\infty} ar^{k-1+1} = \sum_{k=0}^{\infty} ar^{k} = \sum_{k=2}^{\infty} ar^{k-2}.$$

If you subtract/add from the index, wherever there the index is, you add/subtract—that is,

$$\sum_{k=0}^{\infty} ka_k r^k = \sum_{k=3}^{\infty} (k-3)a_{k-3} r^{k-3}.$$

NOTE In general, ~~$k \in \mathbb{Z}^+ = \{0,1,2,3,\dots\}$~~. $k \in \mathbb{Z}^+ \cup \{0\}$ [RP]. However, for this book, they do not use double stamped letters; they simply define it. Rules can be altered to go negative as long as the rules are not broken. You may see this in differential equations or partial differential equations and other higher math courses. At this level, you should remain in the set provided.

$\mathcal{F}_2$ [Formulate] We found the formula, and the book provided a sort of reverse engineering approach. This is a question that should not be done by cheating with AI or YouTube or Private Tutors. You brain is your resource and the book the is the tool. No one is going to hire an expert as using AI like an uneducated consumer.

NOTE At this point, you would have read the book and have some intuition on what to do. Clearly, I know how to do this already. However, you would be able to figure it out quite easily and quickly if you read the book thoroughly. If you copout and use AI or anything else, just leave math already. If you cannot figure out, take a break and come back. If you still cannot figure out, talk with your professor and no one else. It gets easier!

TYPO ↓ [There is a slight arithmetic typo. We must redo from the beginning.]

Now, Let

$$0.\overline{03} = 0.030303\ldots = 0.03 + 0.0003 + 0.000003 + \cdots$$

$$= \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$$

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$$= \frac{3}{10}\left[\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right] = \frac{3}{10}\left[\sum_{k=1}^{\infty} \frac{1}{10^k}\right] = \frac{3}{10}\left[\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k\right] \neq \sum_{k=1}^{\infty} ar^{k-1}.$$

[LEAVE CLASS] If you use the above technique and it is not allowed even though it seems like it is, the author did not say you can change the index—yet. If you go back into the book, you will see that they solved this problem with the partial sums technique. I will finish the geometric series solution above after we do it the way they want it in the book. Because, if you use AI or YouTube or any other cheating platform, you may not get the appropriate or allowed method, and you will be asked to leave class for plagiarism and cheating.

PARTIAL SUMS OF A GEOEMTRIC SEQUENCE

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}, \qquad r \neq 1$$

$$S_n = a\frac{1 - r^n}{1 - r}, \qquad S_n \to \frac{a}{1 - r} \text{ as } n \to \infty \Rightarrow S = \frac{a}{1 - r}.$$

We already found,

$$0.030303 \ldots = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots = \frac{3}{10^2}\left[1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right]$$

$$= \frac{3}{10^2}\left[1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right],$$

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots = a[1 + r + r^2 + r^3 + \cdots]$$

$$\Rightarrow \quad a = \frac{3}{100}, \qquad r = \frac{1}{10}.$$

Now, from the infinite case, we make the connection after reviewing the book that

$$S_n = a\frac{1 - r^n}{1 - r} \quad \Rightarrow \quad S = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1 - r}.$$

Thus,

$$S = \frac{\frac{3}{100}}{1 - \frac{1}{10}} = \frac{1}{30} \neq \frac{1}{33}$$

TYPO ↑ [We start at the typo next.] The math is sound but there is a slight typo.

Now, Let

$$0.\overline{03} = 0.030303\ldots = 0.03 + 0.0003 + 0.000003 + \cdots$$

$$= \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$$

Correction:

$$= \frac{3}{100} + \frac{3}{10000} + \frac{3}{1000000} + \cdots = \frac{3}{10^2} + \frac{3}{10^4} + \frac{3}{10^6} + \cdots$$

$$= \frac{3}{10^2} + \frac{3}{10^4} + \frac{3}{10^6} + \cdots = \frac{3}{10^2}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right]$$

$$= \frac{3}{10^1}\left[\frac{1}{10} + \frac{1}{10^3} + \frac{1}{10^5} + \cdots\right]$$

Factoring is not working. If it helps, you can use a substitution such as letting $u = \frac{1}{10^2}$. Then,

$$= \frac{3}{10^2} + \frac{3}{10^4} + \frac{3}{10^6} + \cdots = 3\left[\frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \cdots\right]$$

$$= 3\left[\left(\frac{1}{10^2}\right)^1 + \left(\frac{1}{10^2}\right)^2 + \left(\frac{1}{10^2}\right)^3 + \cdots\right] \equiv 3[u^1 + u^2 + u^3 + \cdots]$$

$$= 3\left(\frac{1}{10^2}\right)^1\left[1 + \left(\frac{1}{10^2}\right)^{2-1} + \left(\frac{1}{10^2}\right)^{3-1} + \cdots\right] = \frac{3}{100}\left[1 + \left(\frac{1}{10^2}\right)^1 + \left(\frac{1}{10^2}\right)^2 + \cdots\right]$$

Thus,

$$r = \frac{1}{100}, \qquad a = \frac{3}{100}, \qquad S = \frac{\frac{3}{100}}{1 - \frac{1}{100}} = \frac{1}{33}.$$

$\mathcal{F}_3$ [Finalize] For full credit, one would say something along the lines of:

Therefore, by the partial sums of geometric sequence and infinite summation,

$$S = \frac{1}{33}.$$

[RP] [FC] For you guys, prove or show that this answer is correct and also see if you can use the geometric series within the bounds of the rule of the textbook (if you have it.)

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NOTE On the exam, had we use the Geometric Series with alterations to the indices, we may have gotten zero credit as the book did not use this method. If your professor used it in class, that would change things. It is not necessarily an incorrect approach; it just isn't what was shown in the reading. Meaning, it may not be allowed to be used or used yet.

EXAM SOLUTION METHOD *This should be written out with pencil and paper*

Question

Express the repeating decimal as a fraction.

$$0.030303\ldots$$

Student Response for Full Credit

By Geo Series/Seq,

$$S_n = a\frac{1-r^n}{1-r}, \qquad r \neq 1.$$

Then,

$$0.030303\ldots = \frac{3}{100} + \frac{3}{10000} + \frac{3}{1000000} + \cdots = \frac{3}{10^2} + \frac{3}{10^4} + \frac{3}{10^6} + \cdots$$

$$= \frac{3}{10^2} + \frac{3}{10^4} + \frac{3}{10^6} + \cdots = \frac{3}{10^2}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right] = \frac{3}{10^1}\left[\frac{1}{10} + \frac{1}{10^3} + \frac{1}{10^5} + \cdots\right]$$

Factoring is not working. If it helps, you can use a substitution such as letting $u = \frac{1}{10^2}$.

Then,

$$= \frac{3}{10^2} + \frac{3}{10^4} + \frac{3}{10^6} + \cdots = 3\left[\frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \cdots\right]$$

$$= 3\left[\left(\frac{1}{10^2}\right)^1 + \left(\frac{1}{10^2}\right)^2 + \left(\frac{1}{10^2}\right)^3 + \cdots\right] \equiv 3[u^1 + u^2 + u^3 + \cdots]$$

$$= 3\left(\frac{1}{10^2}\right)^1\left[1 + \left(\frac{1}{10^2}\right)^{2-1} + \left(\frac{1}{10^2}\right)^{3-1} + \cdots\right] = \frac{3}{100}\left[1 + \left(\frac{1}{10^2}\right)^1 + \left(\frac{1}{10^2}\right)^2 + \cdots\right]$$

Thus,

$$r = \frac{1}{100}, \qquad a = \frac{3}{100}, \qquad S = \frac{\frac{3}{100}}{1 - \frac{1}{100}} = \frac{1}{33}.$$

Prove 1 Does Not Equal 0 Under Stewart PreCalculus Algebraic Operations

The problem amateur mathematicians have when doing this proof, is they miss half of the summation, which leads to $1 - 1 = 0 \Leftrightarrow 1 = 0$. **This is wrong**.

Proof

Let $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3 \dots\}$ (the integer set).

Let $0 = 0$.

Then,

$$1 - 1 = 0$$

$$\Rightarrow \quad 1 + (-1) = 0 \text{ [additive inverse property]}$$

$$\therefore 1 \neq 0.$$

(Q.E.D.) ■

Proof

Let $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3 \dots\}$ (the integer set).

Let $0 = 0$.

Then,

$$1 - 1 = 0$$

$$\Rightarrow \quad 1 + (-1) = 0 \text{ [additive inverse property]}$$

$$\Rightarrow \quad [1 + (-1)] + [1 + (-1)] + [1 + (-1)] + \cdots = 0$$

$$\Rightarrow \quad (1 + 1 + 1 + \cdots + 1) + (-1 - 1 - 1 - \cdots - 1) = 0$$

$$\sum_{n=0}^{\infty} 1 + \sum_{n=0}^{\infty} (-1) = \infty - \infty = ? \text{ [this is not limits so indeterminate does not apply]}$$

The other way is alternating series

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots = 0$$

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$$\sum_{n=0}^{\infty}(-1)^n \neq 0. \qquad \textbf{NOTE}\sum_{n=0}^{\infty}(-1)^n = \text{diverges.}$$

This is where people mess. We are not going out to just any number nor are we going to infinity.

We are going in pairs of $1 + (-1)$. Thus, ∞ is not correct.

$$\sum_{n=0}^{\text{even}}(-1)^n = 1, \qquad \sum_{n=0}^{\text{odd}}(-1)^n = 0$$

If we add to an even number, we get 1 but to an odd, we get 0.

If we are going to infinity, then we have to say this:

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots = \sum_{n=0}^{\infty}[(1) + (-1)] = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}[(1)] + \sum_{n=0}^{\infty}[(-1)] = \infty - \infty = ?$$

Is $\infty - \infty = 0$? In this case, we can define $\infty - \infty \equiv 0$ under the condition that a summation can be grouped to show the nth term is zero, then $\infty - \infty = 0$.

Conjecture *The Tullis-Aker Principle of Summation*

$$\infty - \infty \equiv 0 \Leftrightarrow \sum_{n}[a_n + b_n] = \sum_{n}[0] = 0.$$

E.g.,

$$\sum_{n=0}^{\infty}[(1) + (-1)] = \sum_{n=0}^{\infty}[0] = 0, \qquad \sum_{n=0}^{\infty}[(1)] + \sum_{n=0}^{\infty}[(-1)] = \infty - \infty.$$

Thus, by the Tullis-Aker principle in summation analysis, $\infty - \infty = 0$.

(Q.E.D.) ■

The Cauchy Principal is similar to what I just defined where the statement for something such as

$$\int_{-\infty}^{\infty}\frac{1}{x}dx = \int_{-\infty}^{0}\frac{1}{x}dx + \int_{0}^{\infty}\frac{1}{x}dx = \infty - \infty \overset{\rightarrow}{\equiv} 0.$$

What is happening here: Mathematically speaking, the answer is "does not converge." However, for application, the difference is so minute that it can be ignored to say that it is zero. However, this is a different subject referencing convergence and divergence with limits. We are not using limits, so it does apply to our arithmetic approach.

[RP] For the PLEM Academy students, you can review this document, and I encourage you to find the broken proof that leads to $1 = 0$. You can find this in some textbooks or via idiots on YouTube. Go find the broken proof and prove that it is not true.

[HINT] The broken proof suggests

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots = 0$$

$$\Rightarrow \quad 1 + (-1) - (-1) + (-1) - (-1) + \cdots = 1$$

You will end up with two summations where one is zero and the other is one but it easy to miss one of them.

For "**Conjecture** *The Tullis-Aker Principle of Summation*," prove the following

$$\sum_k [a_k + b_k] = 0 \Leftrightarrow c_k = a_k + b_k = 0.$$

4.5 Newton's Third Law

QUESTION [3] 4.25

A student of mass 45 kg jumps off a high diving board. What is the acceleration of the earth toward her as she accelerates toward the earth with an acceleration of $9.8\ \mathrm{m/s^2}$? Use 6.0×10^{24} kg for the **mass** of the earth, and assume that the **net force** on the earth is the **force** of **gravity** she exerts on it.

ANSWER

$$7.4 \times 10^{-23}\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

SOLUTION

- Identify Significant Figure There are two significant figures.
- Identify **Keywords**: gravity, force, net force, acceleration

[1st – Extract Data] 50% of the credit on exams.

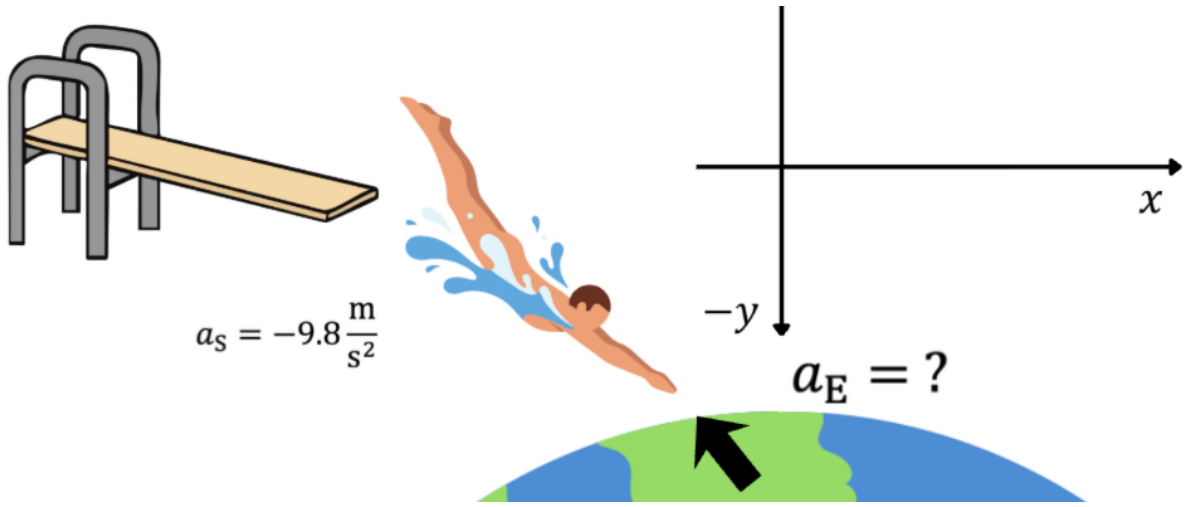
Given (directly give)	**Need** (indirectly give)	**Seeking**

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$m_S = 45\ \text{kg}, \quad a_S = -9.8\frac{\text{m}}{\text{s}^2}$ $m_E = 6.0 \times 10^{24}\ \text{kg}$	$F_{S\text{ on }E} = -F_{E\text{ on }S}$ $F = ma$	acceleration of the earth toward her $a_E = ?$

• Convert Units

[2nd – Formulate Data] 50% of the credit on exams.



We have a simple equation to work with, $F_{S\text{ on }E} = -F_{E\text{ on }S}$.

$$F_{S\text{ on }E} = -F_{E\text{ on }S} \quad \Rightarrow \quad m_S a_S = -m_E a_E \quad \Rightarrow \quad a_E = -\frac{m_S a_S}{m_E}$$

$$\Rightarrow \quad a_E = -\frac{(45\ \text{kg})\left(-9.8\frac{\text{m}}{\text{s}^2}\right)}{(6.0 \times 10^{24}\ \text{kg})} = 7.35 \times 10^{-23}\ \frac{\text{m}}{\text{s}^2}.$$

[3rd – State the Answer] 0% of the credit on exams (but no credit without it).

Therefore, by Newton's Third Law,

$$a_E = 7.4 \times 10^{-23}\ \frac{\text{m}}{\text{s}^2}.$$

[PHFC] How do we know for sure that this is correct? We can take it on good faith, that successful experiment shows that the equation holds in real life application.

The mathematical full circle, is to simply balance the original equation—that is,

$$F_{\text{S on E}} = -F_{\text{E on S}} \;\Leftrightarrow\; m_{\text{S}}a_{\text{S}} = -m_{\text{E}}a_{\text{E}}$$

$$\Leftrightarrow\; (45\text{ kg})\left(-9.8\frac{\text{m}}{\text{s}^2}\right) = -(6.0\times10^{24}\text{ kg})\left(7.35\times10^{-23}\ \frac{\text{m}}{\text{s}^2}\right)$$

$$\Leftrightarrow\; -441\text{ N} = -441\text{ N} \;\Leftrightarrow\; 0 = 0.$$

Thus, the full circle of math is complete. Now, you can do an experiment to see if it works in real life. That is the real test of an experimental physicist.

EXAM Writing (shorthand) (This should be done with pen and paper.)

$$(\textbf{Step 1})\ m_{\text{S}} = 45\text{ kg}, \qquad a_{\text{S}} = -9.8\frac{\text{m}}{\text{s}^2}, \qquad m_{\text{E}} = 6.0\times10^{24}\text{ kg}, \qquad a_{\text{E}} = ?,$$

$$F_{\text{S on E}} = -F_{\text{E on S}}.$$

$$(\textbf{Step 2})\ m_{\text{S}}a_{\text{S}} = -m_{\text{E}}a_{\text{E}} \;\Rightarrow\; a_{\text{E}} = -\frac{m_{\text{S}}a_{\text{S}}}{m_{\text{E}}} \;\Rightarrow\; a_{\text{E}} = -\frac{(45\text{ kg})\left(-9.8\frac{\text{m}}{\text{s}^2}\right)}{(6.0\times10^{24}\text{ kg})} = 7.35\times10^{-23}\ \frac{\text{m}}{\text{s}^2}.$$

(**Step 3**) Therefore, by Newton's Third Law, $a_{\text{E}} = 7.4\times10^{-23}\ \frac{\text{m}}{\text{s}^2}$.

Integrating the Dirac Delta Function over all Space

[12] 15.(c) Use (11.6) and (11.14) to (11.16) to evaluate the following integrals. Warning hint: See comments just after (11.6) and (11.16) about the range of integration.

Dirac Delta Function

$$\int_{\mathbb{R}} \phi(x)\delta^{(n)}(x-a)\,dx = (-1)^n\phi^{(n)}(a)$$

Evaluate the following integral,

$$\int_{-1}^{1} e^{3x}\delta'(x)\,dx.$$

First get it to match the formula.

$$\int_{-1}^{1} e^{3x}\delta'(x)\,dx = \int_{-1}^{1} e^{3x}\delta^{(1)}(x-0)\,dx$$

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For this situation, we are not integrating over all space. So we look to another definition or theorem.

The boundaries for

$$\int_{-\infty}^{\infty} \phi(x)\delta^{(n)}(x-a)\,dx$$

$$= \int_{-\infty}^{-1} \phi(x)\delta^{(n)}(x-a)\,dx + \int_{-1}^{1} \phi(x)\delta^{(n)}(x-a)\,dx + \int_{1}^{\infty} \phi(x)\delta^{(n)}(x-a)\,dx$$

$$= 0 + \int_{-1}^{1} \phi(x)\delta^{(n)}(x-a)\,dx + 0$$

$$= \int_{-1}^{1} e^{3x}\delta^{(1)}(x-0)\,dx = (-1)^1 \frac{d}{dx} e^{3x}\Big|_{x=1} = -3e^{3x}.$$

Is this conclusion correct? The question told us to use (11.6) and (11.14) to (11.16). Now, you can go to the book and think and come to a conclusion of whether or not this is correct after reading.

I will come back to this after you all review it and put your input in.

Motion in Two or Three Dimensions: Relative Velocity

QUESTION [3] 3.37

A canoe has a velocity of 0.40 m/s southeast relative to the earth. The canoe is on a river that is flowing 0.50 m/s east relative to the earth. Find the velocity (magnitude and direction) of the canoe relative to the river.

ANSWER

$$0.36\ \frac{\text{m}}{\text{s}}, \qquad 52.5° \text{ south of west}$$

SOLUTION

NOTE You would have read the chapter before lecture and solved the example questions prior to lecture. You would have taken notes about what you are confused about for the professor and lecture prior to class.

“The biggest mistake a college student makes when solving a physics problem, is reading the question.”—AuthorJonD (Read the question after extrapolation of the proper data.)

• Identify **Significant Figures**:
• Identify **Keywords**:

[1st – Extract Data] 50% of the credit on exams.

Given (directly give)	**Need** (indirectly give)	**Seeking**

• Convert Units

[2nd – Formulate Data] 50% of the credit on exams.

[3rd – State the Answer] 0% of the credit on exams (but no credit without it).

EXAM SOLUTION

Differential Equations vs Physics Equations

Constant Acceleration. Let us begin with the letter a for acceleration. We will then take the antiderivative of acceleration.

$$\frac{dv}{dt} = a \quad \Rightarrow \quad dv = adt \quad \Rightarrow \quad \int dv = \int a\,dt$$

$$\Rightarrow \quad v = at + v_0 \quad \equiv \quad y = ax + C \quad \equiv \quad dy = adx \quad \equiv \quad y + c_1 = ax + c_2, \qquad c_2 - c_1 = c_3 \equiv C.$$

The initial acceleration equation $v'(t) = a$ is a first-order-linear-nonhomogeneous ODE (by definition).

$$v'(t) \equiv \dot{v} = \frac{dv}{dt}$$

$$\frac{d}{dt}v = v^{(1)} = v' \neq v^1$$

Continuing with physics notation related to calculus—integration, we derive the kinematic equation motion.

$$v = at + v_0 \quad \Rightarrow \quad \frac{ds}{dt} = at + v_0 \quad \Rightarrow \quad ds = (at + v_0)dt$$

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$$\Rightarrow \quad \int ds = \int at + v_0\, dt \quad \Rightarrow \quad s = \frac{1}{2}at^2 + v_0 t + s_0 \quad \equiv \quad \Delta x = \frac{1}{2}a_x t^2 + v_{0,x} t.$$

To relate to differential equations:

$$a \equiv \frac{dx^2}{dt^2}, \qquad v \equiv \frac{dx}{dt}, \qquad x \equiv s.$$

In relation to a standard ODE, the kinematic equation is nothing more than a second-differential equation, as follows.

$$s = \frac{1}{2}at^2 + v_0 t + s_0 \quad \equiv \quad y^{(0)}t^0 = \frac{1}{2}y''t^2 + y't + s_0 \quad \equiv \quad \frac{1}{2}\ddot{x}t^2 + \dot{x}t - x = x_0.$$

At first, you will see in ODE, a second order differential equation with a solution method. I want to emphasize the importance of the notation. Because, as you do physics, your brain is going to latch to techniques you used with basic algebra when you need to make the connection of differential equation methos.

When you see a kinematic equation such as $\Delta x = \frac{1}{2}a_x t^2 + v_{0,x} t$, your mind probably jumps to plugging numbers in. However, when you are in an ODE or PDE course, you will need to apply those techniques as you will have the numbers you think you should have.

For this second order ODE, $y = \frac{1}{2}y''t^2 + y't + s_0$, it is called the Cauchy-Euler technique or method. You would solve it in physics 1 just plugging numbers in. In ODE, you will use a technique to solve the differential equation for all cases.

Introduction to Polynomial Vector Notation in Linear Algebra

$$\mathcal{P} = \left\{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \middle| a \in \mathbb{R}, n \in \mathbb{Z}^+ \cup \{0\}\right\}$$

Let's look at a basic quadratic equation in terms of vector notation.

$$f(x) = x^2 - x + 1 = (1)(x^2) + (-1)(x^1) + (1)(x^0) = \begin{bmatrix} x^2 & x & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}.$$

Since we are usually seeking column vectors, $\begin{bmatrix} x^2 & x & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, would be the better choice.

We can now rewrite the vector with respect to the basis, $\mathbb{P}^2$. The following notations may vary book to book or course to course. Make sure to check your textbook and ask your professor which notation is preferred. I will use $\mathcal{P}_n$ with the index implying the number of objects in the set vs. the exponent of the leading term.

$\mathbb{P}^n \equiv \mathbb{P}_n$	$\mathcal{P}^n \equiv \mathcal{P}_n$	$\mathbb{P}^n \equiv \mathcal{P}_{n+1}$
$\mathbb{P}^2 = \{1, x, x^2\} = \mathbb{P}_2$	$\mathcal{P}^2 = \{1, x\} = \mathcal{P}_2$	$\mathbb{P}^2 = \{1, x, x^2\} = \mathcal{P}_3$

$$\therefore f(x) = x^2 - x + 1 = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{P}_3} = \mathbf{f}.$$

We now have the vector notation, $\mathbf{f} = (1, -1, 1)_{\mathcal{P}_3}$. Now, let's look at this in terms of matrices. Determine if the set, $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \{1 + t, t - 1, t^2\}$ is linearly dependent. Do the math two different ways: (1) by spanning the set and seeing if the trivial solution is the only solution, and (2) by inspection.

$$\text{span}\{1 + t, t - 1, t^2\} = (1 + t)x_1 + (t - 1)x_2 + (t^2)x_3$$

$$= \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_3$$

$$= \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_3 \right] = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathcal{P}_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \underset{R_1 + (-1)R_2 \Leftarrow R_2}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathcal{P}_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We can now see that the vectors are linearly independent, since the matrix is full rank.

Motion in Two or Three Dimensions: Relative Velocity

QUESTION [3] 3.37

A canoe has a velocity of 0.40 m/s southeast relative to the earth. The canoe is on a river that is flowing 0.50 m/s east relative to the earth. Find the **velocity** (magnitude and direction) of the canoe relative to the river.

ANSWER

$$0.36 \frac{\text{m}}{\text{s}}, \qquad 52.5° \text{ south of west}$$

SOLUTION

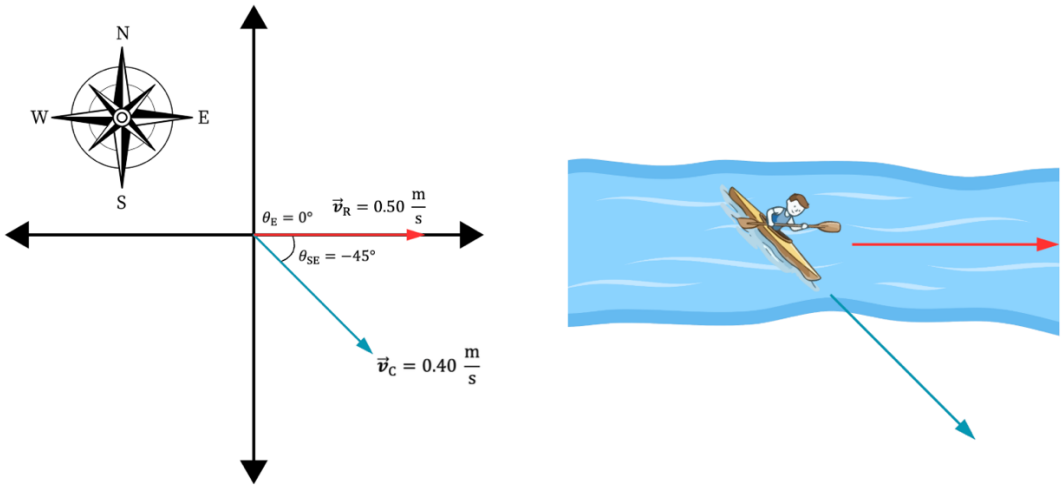
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NOTE You would have read the chapter before lecture and solved the example questions prior to lecture. You would have taken notes about what you are confused about for the professor and lecture prior to class.

"The biggest mistake a college student makes when solving a physics problem, is reading the question."—AuthorJonD (Read the question after extrapolation of the proper data.)

1st Start from the end of the question and read backwards to seek keywords. When you read the question, your ego is going to try and solve it, and it won't do a good job. **This is not magic**; it is an intricate and in-depth language.

2nd You may want to draw a free body diagram. I will do this in Canva. You can use whatever you want. I would recommend using technology and not hand draw or digital draw anything. You need 3-5 years of experience using various software for your resume. Five years of drawing by hand will set you up for a great job as an artist (a bad one).



Free Body Diagram

3rd Read the question.

- Identify **Significant Figures**: 2 Sig Figs
- Identify **Keywords**:

[1st – Extract Data] 50% of the credit on exams.

Given (directly give)	**Need** (indirectly give)	**Seeking**
	Relative velocity along a line $v_{P/A-x} = v_{P/B-x} + v_{B/A-x}$	

$\vec{\boldsymbol{v}}_{\mathrm{C}} = 0.40\ \frac{\mathrm{m}}{\mathrm{s}}, \qquad \theta_{\mathrm{SE}} = -45°$ $\vec{\boldsymbol{v}}_{\mathrm{R}} = 0.50\ \frac{\mathrm{m}}{\mathrm{s}}, \qquad \theta_{\mathrm{E}} = 0°$	$v_{P/A-x}$ x-velocity of P relative to A $v_{P/B-x}$ x-velocity of P relative to B $v_{P/A-x}$ x-velocity of B relative to A	$\lvert\vec{\boldsymbol{v}}_{\mathrm{C}}\rvert = ?$ $\theta = ?$

• Convert Units: Nothing to convert for this question.

TYPO ↓ We referenced the wrong frame of reference and example.

[TTP] **TEST TAKING TIP** EXAMPLE 3.14, "Flying in a crosswind" from the reading is essentially the same question. If this question were assigned for the homework, then the question the reading, would likely be on the test. However, the question in the reading has right triangle or perpendicular(orthogonal) relation whereas our question does not.

[2nd – Formulate Data] 50% of the credit on exams.

For this question, we need to break down the components into x and y.

$$v_{\mathrm{C}-x} = \vec{\boldsymbol{v}}_{\mathrm{C}} \cos\theta_{\mathrm{C}} = 0.40\cos(315°)\ \frac{\mathrm{m}}{\mathrm{s}}, \qquad v_{\mathrm{C}-y} = 0.40\sin(315°)\ \frac{\mathrm{m}}{\mathrm{s}},$$

$$v_{\mathrm{R}-x} = 0.50\cos(0°)\ \frac{\mathrm{m}}{\mathrm{s}}, \qquad v_{\mathrm{R}-y} = 0.50\sin(0°)\ \frac{\mathrm{m}}{\mathrm{s}}.$$

Based on the formula, $v_{P/A-x} = v_{P/B-x} + v_{P/A-x}$. It tells me to add the x and then add the y and then use Pythagorean's Theorem (I think).

$$v_{P/A-x} = v_{P/B-x} + v_{B/A-x}$$

$$v_{\mathrm{R}-x} = v_{\mathrm{C}-x} + v_{\mathrm{R}-x} = 0.40\cos(315°)\ \frac{\mathrm{m}}{\mathrm{s}} + 0.50\cos(0°)\ \frac{\mathrm{m}}{\mathrm{s}}$$

$$v_{\mathrm{R}-y} = v_{\mathrm{C}-y} + v_{\mathrm{R}-y} = 0.40\sin(315°)\ \frac{\mathrm{m}}{\mathrm{s}} + 0.50\sin(0°)\ \frac{\mathrm{m}}{\mathrm{s}}$$

Based on the textbook's solution, we now just need to apply Pythagoras's Theorem—that is,

$$v = \sqrt{\left(0.40\cos(315°)\ \frac{\mathrm{m}}{\mathrm{s}} + 0.50\cos(0°)\ \frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \left(0.40\sin(315°)\ \frac{\mathrm{m}}{\mathrm{s}} + 0.50\sin(0°)\ \frac{\mathrm{m}}{\mathrm{s}}\right)^2}$$

TYPO NOTE First type, I said $360° - 45° = 345°$ [wrong]. It is 315°. The next typo- I forgot the 0°.

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TYPO ↑ We should have used EXAMPLE 3.15 Correcting for a crosswind

Because of the nature of the movement, we should have subtracted the components—not added them as follows.

$$v = \sqrt{\left(0.40\cos(315°)\ \frac{\text{m}}{\text{s}} - 0.50\cos(0°)\ \frac{\text{m}}{\text{s}}\right)^2 + \left(0.40\sin(315°)\ \frac{\text{m}}{\text{s}} - 0.50\sin(0°)\ \frac{\text{m}}{\text{s}}\right)^2}$$

$$\approx 0.356591\ \frac{\text{m}}{\text{s}}.$$

$$\theta = \arctan\left[\frac{\left(0.40\sin(315°)\ \frac{\text{m}}{\text{s}} - 0.50\sin(0°)\ \frac{\text{m}}{\text{s}}\right)}{\left(0.40\cos(315°)\ \frac{\text{m}}{\text{s}} - 0.50\cos(0°)\ \frac{\text{m}}{\text{s}}\right)}\right] = 52.5°$$

$$v_y \approx -0.28, \qquad v_x \approx -0.22.$$

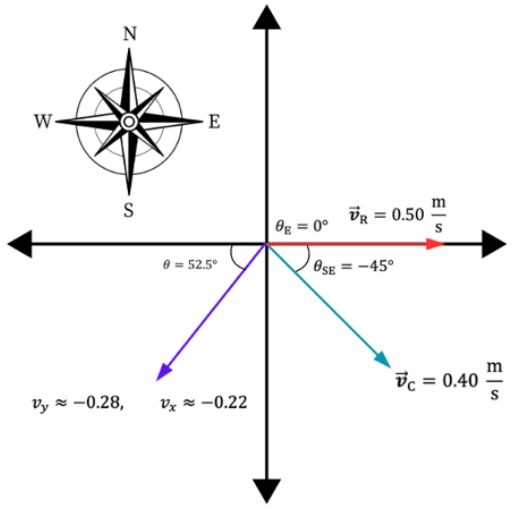
Because coordinates land in Quadrant III (both negative), we know that the angle 52.5°, which is in QI but the coordinates are in QIII and the arctangent function has a domain between $-90°$ and $90°$.

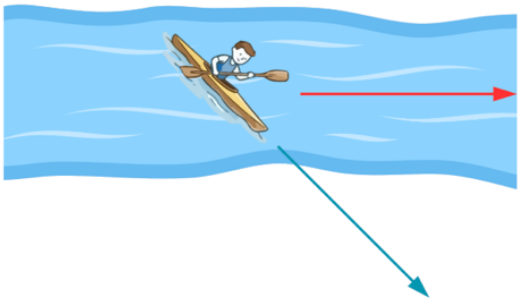
[3rd – State the Answer] 0% of the credit on exams (but no credit without it).

Therefore, by "relative velocity along a straight line," we see that

$$v = 0.36\ \frac{\text{m}}{\text{s}}, \qquad \theta = 52.5°.$$

[RC] Student Name [student who caught the mistake]





EXAM SOLUTION

For the exam, I would shorthand the above information and for the final calculation, I would not plug it into a calculator until I solved every other problem—that is, I would leave the below as the answer until I answer all questions on the exam. If I have time left, I would then go back and plug the following into a calculator.

You should(in most cases) get 100% credit for the answer written next even though you did not approximate to the correct sig figs. Anyone can plug numbers in a calculator. What is important is that you identify everything correctly, label everything, and assemble everything.

Formula ½ credit

$$v_{P/A-x} = v_{P/B-x} + v_{B/A-x} \Rightarrow v_{P/A-x} - v_{B/A-x} = v_{P/B-x}$$

Formulate

$$v_{C-x} = \vec{v}_C \cos\theta_C = 0.40\cos(315°)\ \frac{m}{s}, \qquad v_{C-y} = 0.40\sin(315°)\ \frac{m}{s},$$

$$v_{R-x} = 0.50\cos(0°)\ \frac{m}{s}, \qquad v_{R-y} = 0.50\sin(0°)\ \frac{m}{s}.$$

Answer

$$v = \sqrt{\left(0.40\cos(315°)\ \frac{m}{s} - 0.50\cos(0°)\ \frac{m}{s}\right)^2 + \left(0.40\sin(315°)\ \frac{m}{s} - 0.50\sin(0°)\ \frac{m}{s}\right)^2},$$

$$\theta = \arctan\left[\frac{\left(0.40\sin(315°)\ \frac{\text{m}}{\text{s}} - 0.50\sin(0°)\ \frac{\text{m}}{\text{s}}\right)}{\left(0.40\cos(315°)\ \frac{\text{m}}{\text{s}} - 0.50\cos(0°)\ \frac{\text{m}}{\text{s}}\right)}\right].$$

I would just leave it un-simplified until all questions on the exam answered as the question never specified to write the answer to the correct sig-figs.

NOTE This question and the two examples in the reading cover the three scenarios for this relation. Now, you should beat it into your head the three different cases so when exam comes, you will instinctively know which case to use and it will take you less than 5 minutes to do this problem that took us about 1.5 hours to solve (which would have been 15-30 minutes on your own as I am typing a professional document while explaining.)

[RP] Can you prove this answer is correct and or do an experiment to verify it?

First Order Linear Differential Equations – Integrating Factor Formula

---

QUESTION [32] 1.1.6 & 1.1.7

[1.1.6] Find the solution of the IVP

$$y' = -4y + 2, \qquad y(0) = 5.$$

[1.1.7] Find the solution of the IVP [Solve on your own. I wanted to show the notation.]

$$\frac{dy}{dt}(t) = 3y(t) - 2, \qquad y(1) = 1.$$

ANSWER Pg. 412

[1.1.6]

$$y(x) = \frac{9}{2}e^{-4t} + \frac{1}{2}, \qquad \text{with } c \in \mathcal{R}.$$

[1.1.7]

$$y(x) = \frac{5}{3}e^{3t} + \frac{2}{3}.$$

SOLUTION

NOTE The notation in this book is not standard and has discrepancies—that is, for [1.1.7], we see $\frac{dy}{dt}(t)$. This could be misinterpreted as taking the derivate of y as a product with t. The professor means, $\frac{dy(t)}{dt} = 3y(t) - 2$ not $\frac{dy}{dt}(t) = 3y(t) - 2$ or $y'(t) = 3y(t) - 2$.

[1.1.6]

$\mathcal{F}_1$ Identify Formulae [Formulae]

Before we can solve this, we have to reference the allotted rules within the previous reading permitted to execute an algorithm. We reference the book. We cannot use any techniques outside of section 1.1.1 from the textbook [31].

How do we identify the formulae, definitions, theorems, ...? We look to the instructions, which are in the question(s)–that is, in this case, "Find the solution of the IVP." Since we are not given explicit instructions on what can or cannot use (if it just says, "find the solution," then you can do whatever you want). In this case, the professor still requires you to bound by the previous content and only this content.

Based on the books recommendation for solving the IVP of first-order nature, we should use the follow technique.

THEOREM 1.1.4 (Constant Coefficients IVP) Pg. 10/11

Given the constants $a, b, t_0, y_0 \in \mathbb{R}$, with $a \neq 0$, the initial value problem

$$y' = ay + b, \qquad y(t_0) = y_0,$$

has the unique solution

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a}.$$

NOTE The TA that wrote the answers in the back of the book wrote $y(x)$ not $y(t)$. This is a typo. It should address that identifying typos is part of math-learning. Complaints are not.

$\mathcal{F}_2$ Derive Algorithm [Formulate]

The theorem is the solution. It is the only acceptable solution for this textbook. If you are assigned this book, and you use a different book to solve the problem, you will be asked to leave college (or at least the course).

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Use the following to solve the problem

$$y' = ay + b, \qquad y(t_0) = y_0, \qquad y(t) = \left(y_0 + \frac{b}{a}\right)e^{a(t-t_0)} - \frac{b}{a}.$$

$$y' = -4y + 2, \qquad y(0) = 5.$$

$$\therefore \quad y(t) = \left(y_0 + \frac{b}{a}\right)e^{a(t-t_0)} - \frac{b}{a} = \left(5 + \frac{2}{-4}\right)e^{(-4)(t-0)} - \frac{2}{(-4)} = \left(\frac{10}{2} - \frac{1}{2}\right)e^{-4t} = \frac{9}{2}e^{-4t} + \frac{1}{2}.$$

$\mathcal{F}_3$ Finalize Statement [Finalize]

Thus,

$$y(t) = \frac{9}{2}e^{-4t} + \frac{1}{2}.$$

Is it necessary to address $\mathcal{R}$? The student(TA) who wrote the solution made a huge typo and the questions states the answer is in $\mathbb{R}$ which contradicts $\mathcal{R}$ used for notation consistency.

The Full Circle of Math

For an ODE, the simplest way to show the answer is satisfactory, is to plug it in to the original equation.

Answer

$$y(t) = \frac{9}{2}e^{-4t} + \frac{1}{2}.$$

Question Find the solution of the IVP

$$y' = -4y + 2, \qquad y(0) = 5.$$

Since we have a first order ODE, we can simply plug it into the original differential equation to see if we get $0 = 0$.

$$y(t) = \frac{9}{2}e^{-4t} + \frac{1}{2} \quad \Rightarrow \quad y'(t) = -18e^{-4t}$$

$$\Rightarrow \quad y' = -4y + 2 \quad \Leftrightarrow \quad -18e^{-4t} = -4\left(\frac{9}{2}e^{-4t} + \frac{1}{2}\right) + 2$$

$$\Leftrightarrow \quad -18e^{-4t} = (-4)\frac{9}{2}e^{-4t} + (-4)\frac{1}{2} + 2$$

$$= -18e^{-4t} - 2 + 2$$

$$= -18e^{-4t} + 0$$

$$= -18e^{-4t}.$$

Thus,

$$-18e^{-4t} = -18e^{-4t} \iff 0 = 0.$$

The balance holds implying that the answer satisfies the original question. Thus completing "The Fulle Circle of Math."

It is imperative that you are capable of showing your answers are indeed correct. Puzzle solving by matching numbers is not a skill.

To learn is to choose to understand the truth. To deny the truth is to embed oneself in lies. To lies is to be a non-truth teller. To not tell truth is not mathematics.

Ordinary Differential Equations vs Partial Differential Equations

---

DEFINITION 1

From [33], the author states an ODE is an equation with derivative operations on a scalar function. E.g.,

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = \sin t\,, \qquad x = x(t)$$

is considered to be, 'ordinary.'

Whereas he defines a partial differential equation to be one for which operations are had on more than one variable, such as

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}, \qquad f = u(t, x, y).$$

DEFINITION 2 (I forget which book this is from. I will hunt it down.)

However, the definition does not account for a broader situation. In another textbook, an ODE is described as having one independent variable and a PDE is described as having one independent variable. E.g.,

ODE	PDE
$\frac{dx}{dt} + \frac{dy}{dt} = 0$	$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

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[RP] From the two definitions, is the following an ODE or a PDE?

$$\frac{\partial y}{\partial x} = 1$$

Series Solution ODE

Non-Reference Arithmetic [meaning I am to provide lesson—slang]

To understand is not the objective. To comprehend is a position of thought. To learn, is to continue.

$$y' = 0$$

Solve this

[first classification] first order linear homogeneous ordinary differential equation
[second classification(s)] "constant coefficients," "first-order-linear," "...," ...

[Q] Is $y' = 0$ an autonomous ODE?

QUICKY SOLUTION "Separable Variable"

$$y' = 0 \;\Rightarrow\; \frac{dy}{dx} = 0 \;\Rightarrow\; dy = (0)\,dx \;\Rightarrow\; \int dy = \int (0)\,dx \;\Rightarrow\; y + c_1 = c_2$$

$$\Rightarrow\; y + c_1 = c_2 \;\Rightarrow\; y = c_2 - c_1 = c_3 = C.$$

$$\therefore\; y' = 0 \;\Rightarrow\; y = C.$$

Now, using Series Solution Method to ODE, show the $y' = 0 \;\Rightarrow\; y = C$.

Let,

$$y = \sum_{k=0}^{\infty} a_n (x - x_0)^n.$$

Then,

$$y' = \sum_{k=0}^{\infty} a_n n (x - x_0)^{n-1} = \sum_{k=0+1}^{\infty} a_{n-1} (n-1)(x - x_0)^{n-1+(-1)}$$

$$= \sum_{k=1}^{\infty} a_{n-1} (n-1)(x - x_0)^{n-2}.$$

Assume $x_0 = 0$.

Then,

$$y' = 0 \;\Rightarrow\; \frac{dy}{dx} = \sum_{k=0}^{\infty} a_n n x^{n-1} = \sum_{k=0}^{\infty} [a_n n] x^{n-1} = 0 \;\Rightarrow\; a_n n = 0$$

$a_n n = 0$ is a recursive formula (or is it?)

Clearly, $a_n = 0$. Thus, $a_0 = 0, a_1 = 0, a_2 = 0, \ldots$

Now, let $a_0 = 1, a_1 = 0$ [for first order]
[for second order: let $a_0 = 0, a_1 = 1$ and $a_0 = 1, a_1 = 0$]

Thus,

$$y_1 = \sum_{k=0}^{\infty} a_n (x - x_0)^n = a_0 (x - x_0)^0 + a_1 (x - x_0)^1 + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \cdots$$

$$\Rightarrow\; y = c_1 y_1 \;\Rightarrow\; y = c_1 [a_0 (x - x_0)^0 + a_1 (x - x_0)^1 + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \cdots]$$

$$\Rightarrow\; y = c_1 [(1)(x - x_0)^0 + (0)(x - x_0)^1 + (0)(x - x_0)^2 + (0)(x - x_0)^3 + \cdots]$$

$$\Rightarrow\; y = c_1 [(1)(x - x_0)^0 + (0)(x - x_0)^1 + (0)(x - x_0)^2 + (0)(x - x_0)^3 + \cdots]$$

$$\Rightarrow\; y = c_1 [(1)] = c_1 = C.$$

$$\therefore y = C.$$

Now, show $y' = 0 \;\Rightarrow\; y = C$ using the first order linear homogeneous equation formula.

$$y' + Py = Q, \qquad I = e^{\int P dx}, \qquad y(x) = \frac{1}{I(x)} \left[\int I(x) Q(x)\, dx + C \right].$$

[PL-FAIL] $a(x)y' + P(x)y = Q(x)$ does not work—that is, $y' + P/ay = Q/a$. Coefficient of y' is 1. This is something that students mess up on with reduction of order and other forms.

Thus,

$$y' = y' + (0)y = 0 \;\Rightarrow\; P = 0, \qquad Q = 0, \qquad I = e^{\int 0} = e^{c_1} = c_2.$$

Then,

$$y = \frac{1}{c_2} \left[\int (0)(c_2)\, dx + C \right] \;\Rightarrow\; y = \frac{1}{c_2} [c_3 + C] \;\Rightarrow\; y = \frac{1}{c_2} [c_3 + C] = C' = K.$$

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Let $K \equiv C$, then $y' = C$. Thus, $y' = C$.

RECALL This is an informal non-reference section for slang use. The message is delivered but not formal.

Now, solve the same problem with the constant coefficient method.
Let $y = e^{mx}$. Then, $y' = me^{mx}$.

Then,

$$me^{mx} = 0 \;\Rightarrow\; e^{mx}(m) = 0 \;\Rightarrow\; m = 0, \qquad e^{mx} > 0.$$

NOTE In algebra: $(x-a)(x-b) = 0 \;\Rightarrow\; x - a = 0$ or $x - b = 0$.

In this case, $e^{mx} \neq 0$ nor can it be $e^{mx} < 0$. Thus, $e^{mx} > 0$.

[Q] Would the appropriate English phrase be "$m = 0$ or $e^{mx} > 0$" or "$m = 0$ and $e^{mx} > 0$" or "$m = 0$, $e^{mx} > 0$"?

I am going to say, "$m = 0$ regardless $e^{mx} > 0$.

DEFINE 'regardless' "According to pop-culture (.com's)

A.I.: *adverb* or *adjective* "despite everything," "without paying attention to," or "in spite of" difficulties or objections

Merriam-Webster *adjective* : heedless, careless

Cambridge *adverb* despite; not being affected by something

$$\therefore\; y_1 = e^{(0)x} = e^0 = e^{1-1} = e^1 e^{-1} = e\left(\frac{1}{e}\right) = \frac{e}{e} = 1.$$

$$\therefore\; y = c_1 y_1 = c_1(1) = c_1 = C.$$

$$\therefore\; y = C.$$

Define Insanity: "Seriously mentally ill."—Oxford Languages
Antinsantiy: "Doing the same thing over and over and getting the same results—I.e., ODE."—define by me

---

Now: **Partial Differential Equation** PDE POTENTIAL GIBBERISH ↓

Assume $y' = 0$. Is it true

$$\frac{dy}{dx} = 0 = \frac{\partial y}{\partial x}?$$

You may also then ask

$$\frac{dy}{dx} = 0 = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \quad \Leftrightarrow \quad u_x = u_y \quad \Leftrightarrow \quad u_x = 0 \quad \Leftrightarrow \quad u_y = 0?$$

What does this mean? Essentially, we are **first** in cartesian 2D $\mathbb{R}^2$ (x, y) coordinates. Thus, $y = f(x)$. However, f_x could also be considered as the same as follows:

Let $y = f(x)$. Then, $\frac{\partial}{\partial x} y = \frac{\partial}{\partial x} f(x) = f_x$. Let $u = y = f(x)$. Then, $u_x = f_x$.

The same can be said for $u_y = f_y$.

Thus,

$$\frac{dy}{dx} = 0 = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \quad \Leftrightarrow \quad u_x = u_y \quad \Leftrightarrow \quad u_x = 0 \quad \Leftrightarrow \quad u_y = 0.$$

[Q.PR] **Does the arithmetic violate any rules? If so, which subject are you referring to?**

Okay. So, the above may or may not be correct...it poses a new question:

Let $u = f(x, y)$; solve the PDE,

$$u_y = u_x$$

Before we go forward, does this make sense? Before we go forward, let us consider what we are analyzing.

If $u = f(x, y)$ and $\frac{dy}{dx} = 0 = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$. Then, $y' = f(x) = f_x(x, y) = f_y(x, y)$. Is this acceptable? Are we in the abyss of interdimensional nonsense allowing us to make unorthodox connections? Or is this just nonsense?

[Q] Is this a PDE? Why or why not?

Let us continues,

$$u_y = u_x \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \Rightarrow \partial f \partial y = \partial f \partial x$$

$$\Rightarrow \partial y = \partial x \Rightarrow \int \partial y = \int \partial x \Rightarrow \int dy = \int dx$$

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$$\Rightarrow \quad y + c_1 = x + c_2 \;\Rightarrow\; y = x + c_2 - c_1 = x + c_3 = x + C.$$

Clearly $y = C \neq x$ and $x = C \neq y$ but as previously stated $u_x = 0$ and $u_y = 0$ implying $x = 0$ or $y = 0$. Then, $y = C$ or $x = C$.

THE ABOVE IS POTENTIAL-GIBBERISH ↑.

We conjecture: If $y' = 0$, then

$$\frac{dy}{dx} = 0 = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \quad \Leftrightarrow \quad u_x = u_y \;\Leftrightarrow\; u_x = 0 \;\Leftrightarrow\; u_y = 0.$$

We can easily contradict this to show that it a false statement. But, how?

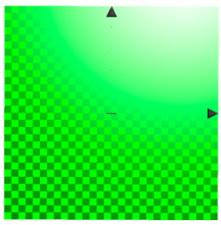
"If the summation of everything is infinity, and the absence of everything is zero. Then zero must be infinity."

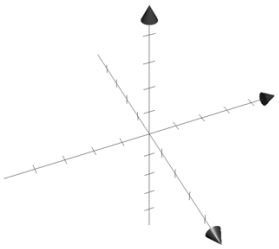
This equation is not accurate because

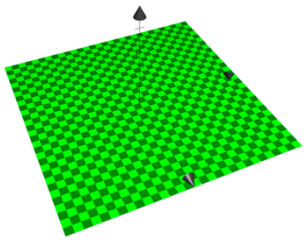
$$\frac{dy}{dx} = 0 = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \;\Rightarrow\; \frac{dy}{dx} = \frac{\partial f}{\partial y}$$

The problem here is that y is the y-axis and f_y

xy-plane







So, if we are in 3D and $f = u(x, y)$ then $z = f(x, y)$ and $f(x, y) = C$. So, $z = C$. Right?

NEXT We are going to relate this to physics. We will depart from the formality of mathematical rule and dive into the 5th dimensional relations of your brain to the visuals you think you are interpreting?...¿

Systems of First-Order Linear Constant-Coefficient Ordinary Differential Equations

QUESTION [23] 6.9.7 Each of the matrices in this problem has some repeated eigenvalues. Determine the general solution to $\dot{\zeta} = A\zeta$ for:

$$A_2 = \begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix}.$$

ANSWER The book does not provide any answers or solutions. We must rely on the full circle of math.

There are a few straight-forward algorithms for solving a system of ODE's. It would be good to look at this deconstructed—that is,

$$\dot{\zeta} = A\zeta \;\Rightarrow\; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\Rightarrow\; \begin{matrix} \dot{x_1} = -3x_1 + x_2 \\ \dot{x_2} = -x_2 - 5x_2 \end{matrix}.$$

We are going to use the textbooks formula as well as Euler's series.

Method 1: Eigen Values and Eigen Vectors

This is pretty easy to do. Just find the Eigenvalues and Eigenvectors

$$\det(A - I\lambda) = \det\left(\begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} -3-\lambda & 1 \\ -1 & -5-\lambda \end{bmatrix}$$

$$= (-3-\lambda)(-5-\lambda) - (1)(-1) = -5(-3-\lambda) - \lambda(-3-\lambda) + 1$$

$$= 15 + 5\lambda + 3\lambda + \lambda^2 + 1 = \lambda^2 + 8\lambda + 16 = (\lambda+4)^2 = 0.$$

The Characteristic Polynomial $f(\lambda) = (\lambda+4)^2$ with multiplicity 2.

$$\therefore \lambda_1 = -4 = \lambda_2.$$

Eigenvectors: Find the nullspace of $[A - I\lambda_i]\mathbf{X} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} -3-\lambda & 1 & 0 \\ -1 & -5-\lambda & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -3-(-4) & 1 & 0 \\ -1 & -5-(-4) & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} x_1 + x_2 = 0 \\ 0 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 + x_2 = 0 \\ x_2 - x_2 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix}$$

$$\Rightarrow \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix} \mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$$

Let $x_2 = 1$. Then, $\mathbf{x} = (-1,1)$. This is one of the eigenvectors. But we need another eigenvector to solve the problem. Maybe even two more.

NOTE When you have unique eigen values and vectors, the answer is simple:

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$$\zeta(t) = c_1\hat{\zeta}^1 e_1^{\lambda_1 t} + c_2\hat{\zeta}^2 e_2^{\lambda_2 t} + \cdots$$

There is a secondary formula for repeated values. I'd like to look at another approach quick.

Method 2: Diagonalization with Euler's Series.

Recall the solution to a basic separable variable equation, $x' = ax$.

$$\frac{dx}{dt} = ax \;\Rightarrow\; \frac{1}{x}dx = adt \;\Rightarrow\; \ln|x| = at + c_1 \;\Rightarrow\; x = e^{at+c_1} = e^{at}e^{c_1} = e^{at}c_2 = ce^{at}.$$

Since

$$e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!}$$

we can apply the same series to the system of differential equations. For a challenge, show the following is true using matrix algebra:

$$\frac{d\mathbf{X}}{dt} = A\mathbf{X} \quad\Rightarrow\quad Ce^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

RESEARCH CHALLENGE Show the following is true with matrix algebra

$$\frac{d\mathbf{X}}{dt} = A\mathbf{X} \quad\Rightarrow\quad Ce^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Thus,

$$\mathbf{X}' = A\mathbf{X} \;\Rightarrow\; \mathbf{X} = C\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

The question is whether or not this will work. We have to diagonalize the matrix to make this work, and since the matrix does not have unique eigenvalues, it is not diagonalizable.

$$C\sum_{k=0}^{\infty} \frac{\begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix}^k t^k}{k!}$$

$$C\sum_{k=0}^{\infty}\frac{A^k t^k}{k!} = C\sum_{k=0}^{\infty}\frac{(SDS^{-1})^k t^k}{k!} = C\sum_{k=0}^{\infty}\frac{SD^k S^{-1} t^k}{k!}$$

$$= CS\left[\sum_{k=0}^{\infty}\frac{D^k t^k}{k!}\right]S^{-1} = CS\left[\sum_{k=0}^{\infty}\frac{\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}^k t^k}{k!}\right]S^{-1}$$

$$= CS\left[\sum_{k=0}^{\infty}\frac{\begin{bmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{bmatrix} t^k}{k!}\right]S^{-1} = CS\left[\sum_{k=0}^{\infty}\frac{\begin{bmatrix}\lambda_1^k t^k & 0\\ 0 & \lambda_2^k t^k\end{bmatrix}}{k!}\right]S^{-1}$$

$$= CS\left[\sum_{k=0}^{\infty}\begin{bmatrix}\frac{\lambda_1^k t^k}{k!} & 0\\ 0 & \frac{\lambda_2^k t^k}{k!}\end{bmatrix}\right]S^{-1} = CS\begin{bmatrix}\sum_{k=0}^{\infty}\frac{\lambda_1^k t^k}{k!} & 0\\ 0 & \sum_{k=0}^{\infty}\frac{\lambda_2^k t^k}{k!}\end{bmatrix}S^{-1}$$

$$= CS\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}S^{-1} = \begin{bmatrix}c_1\\ c_2\end{bmatrix}[\mathbf{v}_1|\mathbf{v}_2]\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}[\mathbf{v}_1|\mathbf{v}_2]^{-1}$$

$$= [c_1 \quad c_2]\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}^{-1}$$

[RP] Multiply the above out to show the following:

$$[c_1 \quad c_2]\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}^{-1} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}.$$

We run into a problem with both methods. The formula within the text is for unique eigenvalues. We must reference the textbook for repeated eigenvalues—that is, a formula. Less we wish to try and discover it ourselves.

From the book, we see for repeated eigenvalues with repeated eigenvectors, the solution for the 2×2 case, is

$$\zeta(t) = c_1\hat{\zeta}^1 e^{\lambda_1 t} + c_2\left(\hat{\zeta}^2 e^{\lambda_1 t} + \hat{\zeta}^1 t e^{\lambda_1 t}\right).$$

In order to find $\hat{\zeta}_2$, we need another formula/equation. From the textbook, we can find $\hat{\zeta}^2$ by solving

$$A\mathbf{x} = \hat{\zeta}^1.$$

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Thus,

$$\left[\begin{array}{cc|c} -3 & 1 & -1 \\ -1 & -5 & 1 \end{array}\right]$$

Wolfram Alpha Input Eigenvalue {{-3,1},{-1,-5}} $= -4$. RREF {{-3,1,-1},{-1,-5,1}}

$$\left[\begin{array}{cc|c} -3 & 1 & -1 \\ -1 & -5 & 1 \end{array}\right] \sim \left[\begin{array}{c|c} I_2 & \begin{matrix} \frac{1}{4} \\ -\frac{1}{4} \end{matrix} \end{array}\right] \Rightarrow \hat{\zeta}^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\therefore \zeta(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} t e^{-4t} \right).$$

[RP] Is this correct? It is now time for you to show this is correct without technology.

Going back to the summation situation. Let's consider the decomposition

$$A = \begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{X} = C \sum_{k=0}^{\infty} \frac{\begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix}^k t^k}{k!} = C \sum_{k=0}^{\infty} \frac{\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right)^k t^k}{k!}$$

[RP] Show$(SDS^{-1})^k = SD^kS^{-1}$

$$= C \sum_{k=0}^{\infty} \frac{\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right)^k t^k}{k!} = C \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{\begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}^k t^k}{k!} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

We as ourselves, what is $\begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}^k$? Because $\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}^k = \begin{bmatrix} a_{11}^k & 0 \\ 0 & a_{22}^k \end{bmatrix}$.

$$\begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}^k = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}^{k-1} = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}^{k-2}$$

$$= \begin{bmatrix} 16 & -8 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}^{k-2} = \begin{bmatrix} 16 & -8 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}^{k-3}$$

[RP] Show the following expansion: Wolfram Alpha expand {{-4,1},{0,-4}}^k

NOTE There is no guarantee Wolfram Alpha is correct.

$$\begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}^k = \begin{bmatrix} (-4)^k & -(-1)^k 4^{k-1} k \\ 0 & -4^k \end{bmatrix}$$

$$= C \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{\begin{bmatrix} (-4)^k & -(-1)^k 4^{k-1} k \\ 0 & -4^k \end{bmatrix} t^k}{k!} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= C \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(-4)^k t^k}{k!} & \sum_{k=0}^{\infty} -\frac{(-1)^k 4^{k-1} k t^k}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{(-4)^k t^k}{k!} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= [c_1 \quad c_2] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-4t} & te^{-4t} \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \left(c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} e^{-4t} + te^{-4t} & te^{-4t} \\ e^{-4t} & e^{-4t} \end{bmatrix}$$

$$= \left(\begin{bmatrix} c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \right) \begin{bmatrix} e^{-4t} + te^{-4t} & te^{-4t} \\ e^{-4t} & e^{-4t} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ -c_1 + c_2 \end{bmatrix} \begin{bmatrix} e^{-4t} + te^{-4t} & te^{-4t} \\ e^{-4t} & e^{-4t} \end{bmatrix}$$

$$= c_1 \begin{pmatrix} e^{-4t} + te^{-4t} \\ e^{-4t} \end{pmatrix} + (-c_1 + c_2) \begin{pmatrix} te^{-4t} \\ e^{-4t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} e^{-4t} + te^{-4t} \\ e^{-4t} \end{pmatrix} + (-c_1 + c_2) \begin{pmatrix} te^{-4t} \\ e^{-4t} \end{pmatrix}$$

$$= \begin{matrix} c_1 e^{-4t} + c_1 te^{-4t} + (-c_1 + c_2)(te^{-4t}) \\ c_1 e^{-4t} + (-c_1 + c_2)(te^{-4t}) \end{matrix}$$

$$= \begin{matrix} c_1 e^{-4t} + c_1 te^{-4t} - c_1 te^{-4t} + c_2 te^{-4t} \\ c_1 e^{-4t} + c_1(-te^{-4t}) + c_2 te^{-4t} \end{matrix}$$

$$= \begin{matrix} c_1 e^{-4t} + c_1 te^{-4t} - c_1 te^{-4t} + c_2 te^{-4t} \\ c_1 e^{-4t} - c_1 te^{-4t} + c_2 te^{-4t} \end{matrix}$$

$$= \begin{matrix} c_1 e^{-4t} + c_2 te^{-4t} \\ c_1 e^{-4t} - c_1 te^{-4t} + c_2 te^{-4t} \end{matrix}$$

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$$= \begin{matrix} c_1 e^{-4t} + c_2 t e^{-4t} + c_2(0) \\ c_1 e^{-4t} - c_1 t e^{-4t} + c_2 t e^{-4t} \end{matrix}$$

[RP] At this point, we have a typo in the above arithmetic. You, as a PLEM Academy researcher, can now attempt to find the typo. We will reconvene after the team revies.

Thermodynamics: [17.4] Thermal Expansion

[3] 17.22 A brass rod 185 cm long and 1.60 cm in diameter. What **force** must be applied to each end of the **rod** to prevent it from **contracting** when it is **cooled** from 120.0°C to 10.0°C?

• Identify Significant Figure: The least sig fig count is 3
• Identify **Keywords**: **contracting**, **cooled**, **rod**, **force**

[1st – Extract Data] 50% of the credit on exams.

Given (directly give)	**Need** (indirectly give)	**Seeking**
$L_0 = 185\text{ cm}, \quad d = 1.60\text{ cm}$	$\Delta L = \alpha L_0 \Delta T$	$F = ?$
$T_0 = 120.0°\text{C}, \quad T = 10.0°\text{C}$	$\frac{F}{A} = -T\alpha\Delta T$	
$\Delta T = 10.0°\text{C} - 120.0°\text{C} = -110.0°\text{C}$	$\Delta T = -110.0°\text{C} = -110.0\text{ K}$	
$A = \pi r^2 = \pi\left(\frac{d}{2}\right)^2 = \pi\left(\frac{1.60\text{ cm}}{2}\right)^2$	$\frac{F}{A} = -Y\alpha\Delta T$	
$\alpha = 2.0 \times 10^{-5}\text{K}^{-1} = 2.0 \times 10^{-5}(\text{C}°)^{-1}$	$\Rightarrow F = -AY\alpha\Delta T$	
$Y_{\text{Brass}} = 9.0 \times 10^{10}\text{ Pa}$		

NOTE The *interval* in Kelvin is the same as Celsius

Initial Length / Final Length

α units are K^{-1} or $(\text{C}°)^{-1}$

From [3], "If an object has length L_0 at temperature T_0, then its length L at temperature $T = T_0 + \Delta T$ is

$$L = L_0 + \Delta L = L_0 + \alpha L_0 \Delta T = L_0(1 + \alpha \Delta T) \quad [17.7]$$

Thermal Stress

$$\frac{F}{A} = -Y\alpha\Delta T$$

- F, is the force needed to keep the length of the rod constant.
- A, Cross sectional area
- Y, Young's modulus [for brass 9.0×10^{10}]
- α, Coefficient of linear expansion
- ΔT, Temperature change

[2nd – Formulate Data] 50% of the credit on exams.

Now, in theory, we have

$$F = -AY\alpha\Delta T$$

$$= -\left(\pi\left(\frac{1.60 \text{ cm}}{2} \cdot \frac{1 \text{ m}}{100 \text{ cm}}\right)^2\right)[9.0 \times 10^{10} \text{ Pa}](2.0 \times 10^{-5}\text{K}^{-1})(-110.0)\text{K}$$

$$= -\left(\pi\left(\frac{1.60}{200} \text{ m}\right)^2\right)[9.0 \times 10^{10} \text{ Pa}](2.0 \times 10^{-5}\text{K}^{-1})(-110.0)\text{K} = 39810.3 \text{ N}.$$

[3rd – State the Answer] 0% of the credit on exams (but no credit without it).

Thus,

$$F = 3.98 \times 10^4 \text{ N}.$$

[FC] We do not have an answer for this in the book. How do we verify the work is correct without using a solution manual or A.I.

$$F = -AY\alpha\Delta T \;\Rightarrow\; \Delta T = -\frac{F}{AY\alpha}$$

Of course, the above, will work to verify since it is the same numbers used to find F so it is not going to satisfy our full circle solution. We would need another equation that represents a given measurement with a different measurement that incorporates the found measurement. If the measurement not involved with finding the force comes up correct, then we can be confident our work is correct.

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At this point in the question, my concern is with the temperature logic. The book says the difference of temperature is the same in Celsius as it is Kelvin but our change is negative and Kelvin is greater than zero.

I request A.I.(ChatGPT) to provide an answer and request that it does not provide a solution so I can simply compare: ChatGPT says 3.8×10^4 N, however, our answer is 3.98×10^4 N. Is this good enough? NO!!! We must be certain. It is important to stick with your textbooks measurements as the precision of measurements changes over time and AI is not necessarily drawing from the current or past data—it is not acceptable for accuracy but can help us in our quest.

Continuing the full circle. We should now, consider a different equation that incorporates a constant and our measures that will output a given measurement with our calculated measurement and then we can feel (more) confident that our answer is in fact correct.

IMPORTANT I did not read this section/chapter before attempting this problem, so I am missing key information that points me to the correct solution. One of the problems we encountered was Young's Modulus. I assumed it was universal on all material, and I chose Aluminum instead brass.

HOMEWORK For the student, your job is to find a different formula that you can use to verify the answer is correct. Hint: Possibly a density formula...?

STOP-CONTINUE for Research Position Student's

We will come back to this and find a way to prove whether or not the answer is correct in the next lesson.

Taylor and Maclaurin Series

[7] Use the Taylor series generated by e^x at $x = a$ to show that

$$e^x = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots\right].$$

Taylor Series

$$T_n(x) = \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This is a very basic question. Simply apply the formula.

Let $f(x) = e^x$. Then, $f'(x) = e^x$, $f''(x) = e^x$, ...

Now, expand the series out to $n = 1$.

$$\sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \sum_{n=1}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Recall the set of nth order derivatives, $\{y, y', y''', \dots\} \equiv \{y^{(0)}, y^{(1)}, y^{(2)}, \dots, y^{(n)}\}$.

I.e., the 'zeroth derivative' is simply the non-derivative function or antiderivative of y', provided the constant is zero.

$$= \frac{f^{(0)}(a)}{0!}(x-a)^0 + \sum_{n=1}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \frac{f(a)}{0!}(x-a)^{1-1} + \sum_{n=1}^{n} \frac{f^{(n)}(x-a)}{n!}(x-a)^n$$

$$= \frac{f(a)}{0!}(x-a)^1(x-a)^{-1} + \sum_{n=1}^{n} \frac{f^{(n)}(x-a)}{n!}(x-a)^n$$

$$= \frac{e^a}{1}\frac{(x-a)}{(x-a)} + \sum_{n=1}^{n} \frac{f^{(n)}(x-a)}{n!}(x-a)^n = e^a + \sum_{n=1}^{n} \frac{f^{(n)}(x-a)}{n!}(x-a)^n$$

You can now complete this on your own, and we will move on to something else.

1.1: Real Numbers

Properties of Real Numbers

QUESTION [28] 1.1.13 *Title of Essay*

State the property of real numbers being used.

$$(x + 2y) + 3z = x + (2y + 3z).$$

ANSWER

Associative Property of Addition

SOLUTION

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Introduction to the Essay

$\mathcal{F}_1$ [Formulae, Definition(s), and or Theorem(s)] **50% of Exam Grade** *0% without the latter*:

Look to the book for the proper terminology:

On page 3, we see from the 'Properties of Real Numbers' that the above is the associativity property.

$$(a + b) + c = a + b + c = a + (b + c).$$

Body of the Essay

$\mathcal{F}_2$ [Formulate: Assemble the Previous Data]**50% of Exam Grade** *0% without the former*:

Let $a = x,\ b = 2y,\ c = 3z.$

Then,

$$(x + 2y) + 3z = x + 2y + 3z = x + (2y + 3z).$$

Summarization and citation of Essay

$\mathcal{F}_3$ [Finalize: Clear State the Final Answer with Proper Reference] **0% of Exam Grade**:

Thus, the associative property is being used.

NOTE The previous information may seem like overkill. However, when students get to higher math, they aren't able to comprehend the questions being asked because they don't understand the terminology and how it relates to the math. It is important to absorb this information in-depth early on to engrave into one's psyche for use in higher math.

[FS (further study)] learn and show how the other properties work on page 3 or from whatever book you are using. Keep with the correct notations and definitions per your book

Properties of Real Numbers

QUESTION 1.1.21 *Title of Essay*

Rewrite the expression using the given property of real numbers. Distributive Property,

$$4(A + B) = [\dots].$$

ANSWER

$$4A + 4B$$

SOLUTION

Introduction to the Essay

$\mathcal{F}_1$ [Formulae, Definition(s), and or Theorem(s)] **50% of Exam Grade** *0% without the latter*:

From page 3, we see the following.

$$a(c+b) = a(b+c) = ab + ac = ba + ca = (b+c)a = (c+b)a$$

Body of the Essay
$\mathcal{F}_2$ [Formulate: Assemble the Previous Data]**50% of Exam Grade** *0% without the former*:

$$4(A+B) = 4A + 4B$$

Summarization and citation of Essay
$\mathcal{F}_3$ [Finalize: Clear State the Final Answer with Proper Reference] **0% of Exam Grade**:

Therefore,

$$4(A+B) = 4A + 4B.$$

Arithmetic Operations

QUESTION 1.1.32 *Title of Essay*

Perform the indicate Operations. [I will do (b). You can do (a)]

$$\textbf{(a)}\ \frac{2}{\frac{2}{3}} - \frac{\frac{2}{3}}{2} \qquad\qquad \textbf{(b)}\ \frac{\frac{2}{5}+\frac{1}{2}}{\frac{1}{10}+\frac{3}{15}}$$

ANSWER

This is an even question. We will use Wolfram Alpha to verify our answer.

SOLUTION

Introduction to the Essay
$\mathcal{F}_1$ [Formulae, Definition(s), and or Theorem(s)] **50% of Exam Grade** *0% without the latter*:

We are dealing with fractions. From the book, we know that there is a formula to use

[PL (point loss)] Students often skip lecture and do not read books and try to guess what things are as if someone told them half the alphabet of the English language, gave them a book with the other half of the alphabet, and asked them to write a story using the alphabet. The students do not read the book and attempt to magically discover the missing letters from the alphabet. You cannot intuitively know what someone else invented; you have told what it is. Thus, we look to the book for which formula we can use.

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From page 5, we see the **Properties of Fractions** is what we need.

Body of the Essay

$\mathcal{F}_2$ [Formulate: Assemble the Previous Data]**50% of Exam Grade** *0% without the former*:

For this question, we need property 4

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd} \quad \equiv \quad \frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm cb}{bd}$$

$$\Rightarrow \quad \frac{\frac{2}{5}+\frac{1}{2}}{\frac{1}{10}+\frac{3}{15}}=\frac{\frac{(2)(2)+(1)(5)}{(5)(2)}}{\frac{(1)(15)+(3)(10)}{(10)(15)}}=\frac{\frac{4+5}{10}}{\frac{15+30}{150}}=\frac{\frac{9}{10}}{\frac{45}{150}}$$

Now, we property 2.

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c} \quad \equiv \quad \frac{\frac{a}{b}}{\frac{c}{d}}=\frac{ad}{bc}$$

$$\Rightarrow \quad \frac{9}{10}\cdot\frac{150}{45}=\frac{(9)(150)}{(10)(45)}=\frac{(150)(9)}{(10)(45)}=\frac{150}{10}\cdot\frac{9}{45}=(15)\left(\frac{1}{5}\right)=\frac{15}{3}=3.$$

Summarization and citation of Essay

$\mathcal{F}_3$ [Finalize: Clear State the Final Answer with Proper Reference] **0% of Exam Grade**:

Thus, the answer is

3.

Wolfram Alpha gets the same answer.

NOTE For those reading or watching via the internet, this is an elaborate explanation of how you should read the book and thinking about this type of math early on.

Go through and work many questions from the prerequisite chapter as you can. This type of simple arithmetic and definition reference will come back in a huge way in Junior year. If you are not thoroughly sifting through the data in the book early on, it will come back and bit you on the booty!

STUDENT/READER

At this point, I will move on to chapter 2. I am not trying to teach you anything nor am I trying to help you cheat and plagiarize your homework via YouTube or social media. YouTube is not an educational platform nor is it a legitimate source. Nobody has a responsibility to act or present any information in a scholarly fashion on a platform that is designed to trick college students into defrauding taxpayer dollar by plagiarizing with sources such as Khan Academy.

You are responsible for reading the assigned literation—that is, scientific data. And you are responsible for learning how to interpret and apply scientific data under the guidance of your professor—not plagiarizing companies and content creators via social media.

"You do not go to college not to go to college."—@AuthorJonD

STUDY Go back through and thoroughly read the prerequisite chapters/sections and work the examples and for the prerequisite sections, I do recommend solving as many questions from the question bank as you can. This is not necessary for future chapters/textbooks; it is for the most basic terminology and application for algebra because it is like the foundation of a house—cement. It is hard and thick and takes a long time to relocate and settle.

2.1: Functions: Difference Quotient

QUESTION 2.1.49 *Title of Essay*

Find $f(a)$, $f(a+h)$, and the difference quotient $\frac{f(a+h)-f(a)}{h}$, where $h \neq 0$.

$$f(x) = 3 - 5x + 4x^2$$

ANSWER

$$3 - 5a + 4a^2, \qquad 3 - 5a - 5h + 4a^2 + 8ah + 4h^2, \qquad -5 + 8a + 4h$$

SOLUTION

Introduction to the Essay

$\mathcal{F}_1$ [Formulae, Definition(s), and or Theorem(s)] **50% of Exam Grade** *0% without the latter*:

Before going forward, we need some formulae:

$$\text{If } y = f(x) \text{ and } x = a. \text{ Then, } y = f(a).$$

Body of the Essay

$\mathcal{F}_2$ [Formulate: Assemble the Previous Data]**50% of Exam Grade** *0% without the former*:

[1] $f(a)$

$$f(x) = 3 - 5x + 4x^2 \;\Rightarrow\; f(x = a) = 3 - 5a + 4a^2 = f(a)$$

[2] $f(a + h)$

$$f(x) = 3 - 5x + 4x^2 \;\Rightarrow\; f(x = a + h) = 3 - 5(a + h) + 4(a + h)^2$$

$$= 3 - 5(a + h) + 4(a + h)^2 \text{ [Distribution Property]}$$

$$\text{[PL (point loss)]} = 3 - 5(a + h) + 4(a + h)^2$$

$$\text{[PL (point loss)]} = 3 + (-5)(a + h) + 4(a + h)^2$$

$$= 3 + (-5)a + (-5)h + 4(a + h)^2 = 3 - 5a - 5h + 4(a + h)^2$$

$$3 - 5a - 5h + 4(a + h)(a + h)$$

$$= 3 - 5a - 5h + 4[(a + h)(a + h)]$$

$$= 3 - 5a - 5h + 4[(a + h)a + (a + h)h]$$

$$= 3 - 5a - 5h + 4[(aa + ha) + (ah + hh)] \text{ [Commutative Property]}$$

$$= 3 - 5a - 5h + 4[(a^2 + ah) + (ah + h^2)]$$

$$= 3 - 5a - 5h + 4[a^2 + ah + ah + h^2]$$

$$= 3 - 5a - 5h + 4[a^2 + 2ah + h^2]$$

$$= 3 - 5a - 5h + 4a^2 + 8ah + 4h^2$$

$$= 4h^2 + 4a^2 + 8ah - 5a - 5h + 3$$

[3] [FR (future reference)] $\frac{f(a+h)-f(a)}{h}$ [comes back in calculus]

[CT (critical thinking)] Before redoing all the arithmetic for this problem, recognize that we already did it in [1] and [2]—that is:

$$\frac{f(a + h) - f(a)}{h} = \frac{[3 - 5a - 5h + 4a^2 + 8ah + 4h^2] - [3 - 5a + 4a^2]}{h}$$

[PL] Drill into your head the negative distribution! This is the number one reason most students do not get A's in Calculus 1 and lower courses.

$$= \frac{3 - 5a - 5h + 4a^2 + 8ah + 4h^2 - [3 - 5a + 4a^2]}{h}$$

$$= \frac{3 - 5a - 5h + 4a^2 + 8ah + 4h^2 - 3 + 5a - 4a^2}{h}$$

$$= \frac{-5h^1 + 8ah^1 + 4h^2}{h^1} = -\frac{5h^1}{h^1} + \frac{8ah^1}{h^1} + \frac{4h^2}{h^1} \text{ [Exponential Rules]}$$

$$= -5h^{1-1} + 8a\frac{h}{h} + 4h^{2-1} = -5h^0 + 8a\frac{h}{h} + 4h^1 = -5(1) + 8a(1) + 4h^1$$

$$-5 + 8a + 4h.$$

[CALC] This question will come back in calculus as taking the derivative of f. As follows.

Let $f(x) = 3 - 5x + 4x^2$. Find the derivative with the power rule and the limit definition of a derivative.

$$f'(x) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = \lim_{h\to 0}(-5 + 8a + 4h) = -5 + 8a + 4\lim_{h\to 0}(h) = -5 + 8a + 4(0)$$

$$\Rightarrow \quad f'(x) = -5 + 8a.$$

Now, for the power rule: $y = x^n \;\Rightarrow\; y' = nx^{n-1}$.

$$\frac{d}{dx}f(x) = \frac{df}{dx} = \frac{d}{dx}[3 - 5x + 4x^2] = \frac{d}{dx}3 - 5\frac{d}{dx}x + 4\frac{d}{dx}x^2$$

$$= 0 - 5\frac{dx}{dx} + (4)(2)[x]^{2-1}\frac{d}{dx}x = 5(1) + (4)(2)x\frac{dx}{dx}$$

$$= 5 + 8x(1) = 5 + 8x.$$

$$\frac{d}{dx}f(x)\Big|_{x=a} \;\Rightarrow\; f'(a) = 5 + 8a.$$

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DEFAULT

QUESTION *Title of Essay*

ANSWER

SOLUTION

Introduction to the Essay
$\mathcal{F}_1$ [Formulae, Definition(s), and or Theorem(s)] **50% of Exam Grade** *0% without the latter*:

Body of the Essay
$\mathcal{F}_2$ [Formulate: Assemble the Previous Data]**50% of Exam Grade** *0% without the former*:

Summarization and citation of Essay
$\mathcal{F}_3$ [Finalize: Clear State the Final Answer with Proper Reference] **0% of Exam Grade**:

Complex Solutions [Algebra]

QUESTION [28] 1.7.70 [We are using technology to verify our answer]

Find all solutions of the equation and express them in the form $a + bi$.

$$t + 3 + \frac{3}{t} = 0$$

SOLUTION

$\mathcal{F}_1$ [Formulae] 50% of Exam Credit

$$at^2 + bt + c = 0 \;\Rightarrow\; t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\mathcal{F}_2$ [Formulate]

$$t + 3 + \frac{3}{t} = 0 \;\Rightarrow\; t\left[t + 3 + \frac{3}{t} = 0\right] \;\Rightarrow\; t^2 + 3t + 3 = 0$$

$$a = 1, \qquad b = 3, \qquad c = 3.$$

$$\therefore \quad t = \frac{-3 \pm \sqrt{(3) - 4(1)(3)}}{2(1)} = \frac{-3 \pm \sqrt{3 - 12}}{2} = \frac{-3 \pm \sqrt{-9}}{2}$$

$$= \frac{-3 \pm \sqrt{(-1)3^2}}{2} = \frac{-3 \pm \sqrt{3^2}\sqrt{-1}}{2} = \frac{-3 \pm 3i}{2}.$$

$\mathcal{F}_3$ [Finalize]

Unless asked for set notation,

$$t = \frac{-3 \pm 3i}{2}.$$

Now, in Wolfram Alpha, I copy and paste the equation and request solve

---

TYPO ↑ During the session, it was pointed out that I forget to square b under the radical.

$$\therefore \quad t = \frac{-3 \pm \sqrt{3^2 - 4(1)(3)}}{2(1)} = \frac{-3 \pm \sqrt{9 - 12}}{2(1)} = \frac{-3 \pm \sqrt{(-1)2^2}}{2(1)}$$

$$= \frac{-3 \pm 2\sqrt{-1}}{2} = \frac{-3 \pm 2i}{2}.$$

$\mathcal{F}_3$ [Finalize]

$$\therefore \quad t = \frac{-3 \pm 2i}{2}.$$

Wolfram Alpha Check [NO A.I. – It is not accurate!]

Solve for t, $t + 3 + \frac{3}{t} = 0$

TYPO ↑ During the session, it was pointed out that I forget to square b under the radical.

$$\therefore \quad t = \frac{-3 \pm \sqrt{3^2 - 4(1)(3)}}{2(1)} = \frac{-3 \pm \sqrt{-3}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{3}i}{2}.$$

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Input interpretation:

solve $t+3+\frac{3}{t}=0$ for t

Results: Approximate forms | Step-by-step solution

$$t=-\frac{1}{2}i\left(\sqrt{3}+-3\,i\right)$$

$$t=\frac{1}{2}i\left(\sqrt{3}+3\,i\right)$$

Image from https://www.wolframalpha.com/ for educational purposes

Just for note- if the professor does not specify "simplify" in higher math, don't. This question did not specify "set notation" either. So, the $\pm$ is totally fine.

$\mathcal{F}_3$ [Finalize]

$$\therefore \quad t=\frac{-3\pm\sqrt{3}i}{2}.$$

Factoring Quadratics [Trigonometry]

QUESTION [28] 7.4.42

Solving Trigonometric Equations by **Factoring** Solve the given equation.

$$2\sin^2\theta - \sin\theta - 1 = 0$$

SOLUTION

$\mathcal{F}_1$ [Formulae]

This is a quadratic equation. Thus,

$$a\sin^2\theta + b\sin\theta + c = 0 \;\Rightarrow\; \sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\mathcal{F}_2$ [Formulate]

$$a = 2, \qquad b = -1, \qquad c = -1.$$

$$\therefore \sin\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(1)}}{2(2)} = \frac{1 \pm \sqrt{1+4}}{4}$$

$$= \frac{1 \pm \sqrt{5}}{4}.$$

$$\therefore \sin\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2(2)} = \frac{1 \pm \sqrt{1+8}}{4}$$

$$= \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4} \;\Rightarrow\; \sin\theta = \frac{4}{4} = 1 \text{ or } \sin\theta = \frac{2}{4} = \frac{1}{2}$$

NOTE The questions stated, "factoring." Because of that, we probably just lost 50% of the credit—that is, make sure you read the question thoroughly before solving it.

$$\Rightarrow \theta = \arcsin(1) \text{ or } \theta = \sin^{-1}\left(\frac{1}{2}\right).$$

$\mathcal{F}_3$ [Finalize]

From the Unit Circle, can I answer from memory quick? I can but at this point, since the question never said to simplify, it is solved. Spending 10-minutes more on this to get 1 extra point, is not

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an effective use of time. It is not an effective use of time because that 10-minutes on a more complicated question could equate to 10+ points vs 1+ point.

Wolfram Alpha Answer

Solve for theta, $2\sin^2\theta - \sin\theta - 1 = 0$.

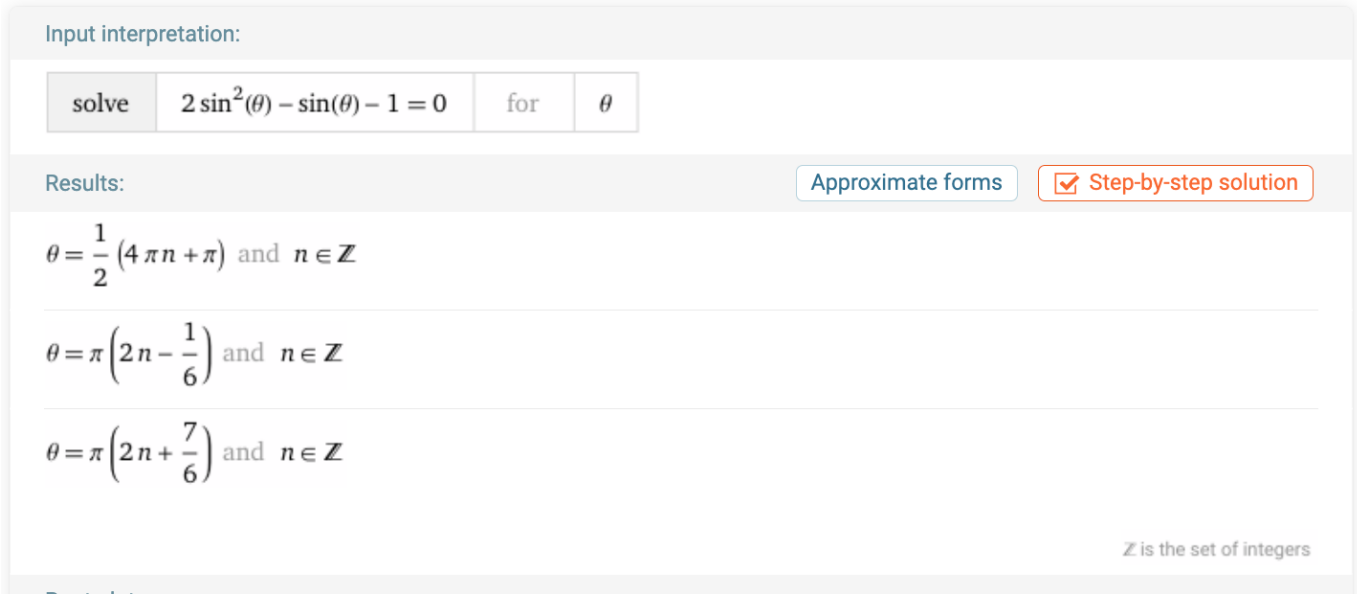


Image from https://www.wolframalpha.com/ for educational purposes

Wolfram Alpha will provide a different answer.

[RP] Take this question and finalize the answer. They did specify and interval, so it should have infinite solutions incorporating n, $n \in \mathbb{N}$ or $n \in \mathbb{Z}^+ \cup \{0\}$, $n \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

QUESTION [28]

Partial Sums of a **Geometric Sequence** Find the sum.

$$\sum_{k=1}^{6} 10(5)^{k-1}$$

SOLUTION

$\mathcal{F}_1$ [Formulae]

$$S_n = \frac{a}{(1-r)}\text{? [missing something]}$$

Student: At this moment, I am not recalling the partial sum formula. I can see it in my head clearly, but when I try to write it down, I forget. This is why you want to practice taking notes and practice exams by hand on demand under stress. That being said, I am going to move to the next question and come back to this after my brain reorganizes itself—that is, I may recall the formula. I just worked with it a few days ago and I can't write it down even though I can see it in my head.

$\mathcal{F}_2$ [Formulate]

$\mathcal{F}_3$ [Finalize]

Moving now to the Stewart Calculus Textbook, [17]

Depending on your University or College, the sequences and series section may be considered Calculus I, II or V (or another).

Limits in Calculus

QUESTION [17] (Professor made question.) **[EGQ]** (Easter Egg Question)

Show with limits the following is true

If

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \Leftrightarrow |r| < 1.$$

Then,

$$\lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} = \frac{1}{1-r}.$$

Easter Egg Question: A question that has the solution within it for another question on the exam that a student may not be able to solve. Meaning, for the previous question on partial sums, $S_n = \frac{a(1-r^n)}{(1-r)}$.

SOLUTION

$\mathcal{F}_1$ [Formulae]

$$\lim_{x\to a} f(x) = f(a)$$

The professor (me) did not specify what technique to use for the limit; use whatever you want provided it comes from a section previous to this section. We are in a end of semester calculus II problem, thus L'Hospital's Rule is acceptable.

$\mathcal{F}_2$ [Formulate]

$$\lim_{n\to\infty}\frac{a(1-r^n)}{(1-r)}=\frac{a(1-r^\infty)}{1-r}=\frac{a(1-\infty)}{1-r}$$

Since it is not in indeterminant form, we will simply use limit laws such as **sum and difference, the exponential, the ratio,** ...

Identify the bold-underlined, is half the credit. The reason why, is because you are writing an English essay. The essay requires citations. Now, some professors may or may not care. Why should they? You are in charge of your education. They are in charge of determining whether or not you took initiation and did it correctly.

$$\lim_{n\to\infty}\frac{a(1-r^n)}{(1-r)}=\frac{a\left(1-\lim\limits_{\mathrm{n}\to\infty}r^n\right)}{(1-r)}=\frac{a\left(1-\lim\limits_{\mathrm{n}\to\infty}r^n\right)}{(1-r)}$$

$$=\frac{\left(a-a\lim\limits_{\mathrm{n}\to\infty}r^n\right)}{(1-r)}=\frac{a}{1-r}-a\lim_{n\to\infty}\frac{r^n}{1-r}$$

Guesswork: At this moment in my head, I know that $\frac{r^n}{1-r}\to 0$ as $n\to\infty$ from experienc. But, my mind tells me it goes to ∞ because of the exponent in the numerator. I am not allowed to use a calculator or have a cheat sheet so I need to think a little differently. I will reference a graph, which can be done by hand quick enough to see where it goes to conclude.

Here is the solution: If $|r|\geq 1$, it is obviously going to infity. But, if $|r|<1$, we see that it goes to zero. I.e. (such that or that is),

$$\frac{a}{1-r}-a\lim_{n\to\infty}\frac{r^n}{1-r}=\infty\Leftrightarrow|r|\geq 1,\qquad\frac{a}{1-r}-a\lim_{n\to\infty}\frac{\frac{1}{r^n}}{1-r}=\frac{a}{1-r}-0\Leftrightarrow|r|<1.$$

$\mathcal{F}_3$ [Finalize]

Therefore,

$$\lim_{n\to\infty}\frac{a(1-r^n)}{(1-r)}=\frac{1}{1-r}.$$

Now, we can go back to the other question after seeing this question because it has the answer to the partial sum in it.

QUESTION [28]

Partial Sums of a **Geometric Sequence** Find the sum.

$$\sum_{k=1}^{6} 10(5)^{k-1}$$

SOLUTION

$\mathcal{F}_1$ [Formulae]

$$S_n = \frac{a}{(1-r)}\text{? [missing something]}$$

Student: At this moment, I am not recalling the partial sum formula. I can see it in my head clearly, but when I try to write it down, I forget. This is why you want to practice taking notes and practice exams by hand on demand under stress. That being said, I am going to move to the next question and come back to this after my brain reorganizes itself—that is, I may recall the formula. I just worked with it a few days ago and I can't write it down even though I can see it in my head.

From the limit question, we see that the partial sum is,

$$S_n = \frac{a(1-r^n)}{(1-r)}, \qquad \sum_{k=0}^{\infty} ar^k = \sum_{k=1}^{\infty} ar^{k-1}$$

$$\sum_{k=1}^{6} 10(5)^{k-1}, \qquad a = 10, \qquad r = 5, \qquad k \in [1,6].$$

$\mathcal{F}_2$ [Formulate]

$$S_k = S_6 = \frac{10(1-5^6)}{(1-5)} = 39060.$$

$\mathcal{F}_3$ [Finalize]

$$\therefore \sum_{k=1}^{6} 10(5)^{k-1} = 39060.$$

Copying $\sum_{k=1}^{6} 10(5)^{k-1}$ into Wolfram Alpha and hitting enter, we get the same answer!

Fundamentals: Real Numbers

QUESTION 1.1.43

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Find the indicated set if $A = \{1,2,3,4,5,6,7\}$, $B = \{2,4,6,8\}$, $C = \{7,8,9,10\}$.

$$(\mathbf{a})\ A \cup C \qquad (\mathbf{b})\ A \cap C$$

ANSWER

$$(\mathbf{a})\ \{1,2,3,4,5,6,7,8,9,10\} \quad (\mathbf{b})\ \{7\}$$

SOLUTION

$\mathcal{F}_1$ [Formula] Let $A = \{a_1.a_2, a_3, \dots\}$, $B = \{b_1, b_2, b_3, \dots\}, a_k \neq b_k\ \forall_k$ (for all \forall ∀)

Intersection, \cap	Union, \cup	Empty Set, \emptyset
$\cap$	$\cup$	$\emptyset$
$A \cap B$	$A \cup B$	
$\{a_1.a_2, a_3\} \cap \{b_1, b_2, b_3, \dots\}$	$\{a_1.a_2, a_3\} \cup \{b_1, b_2, b_3, \dots\}$	
$= \emptyset$	$= \{a_1, b_1, a_2, b_2, \dots\}$	

E.g. (for example), Let $A = \{1,2\}, B = \{2,3\}$.

$$A \cap B = \{1,2\} \cap \{2,3\} = \{2\}, \qquad A \cup B = \{1,2\} \cup \{2,3\} = \{1,2,2,3\} \equiv \{1,2,3\}$$

$\mathcal{F}_2$ [Formulate]

$A = \{1,2,3,4,5,6,7\}$, $B = \{2,4,6,8\}$, $C = \{7,8,9,10\}$.

[a] $A \cup C = \{1,2,3,4,5,6,7\} \cup \{7,8,9,10\} = \{1,2,3,4,5,6,7,7,8,9,10\} = \{1,2,3,4,5,6,7,8,9,10\}$.

[b] $A \cap C = \{1,2,3,4,5,6,7\} \cap \{7,8,9,10\} = \{7\}$.

$\mathcal{F}_3$ [Finalize]

Thus,

$$A \cup C = \{1,2,3,4,5,6,7,8,9,10\}, \qquad A \cap C = \{7\}.$$

NOTE Set $\subset$ \subset (contained) vs element (in \in ∈) of set

$a_1 \in A \equiv a_1 \in \{a_1, a_2, a_3, \dots\}$. (In \in ∈) $\{a_1\} \subset A \equiv \{a_1\} \subset \{a_1, a_2, a_3, \dots\}$.

Section 1.5.13 Uncertainty and Significant Figures

QUESTION A useful and easy-to-remember **approximate value** for the **number of seconds in a year** is $\pi \times 10^7$. Determine the **percent error** in this approximate value. (There are 365.24 days in one year.)

ANSWER

$$0.45\%$$

SOLUTION

• Identify Significant Figure
• Identify **Keywords**: percent error, approximate value, number of seconds in a year

[1st – Extract Data] 50% of the credit on exams.

Given (directly give)	**Need** (indirectly give)	**Seeking**
$t_{\text{total-seconds}} \approx \pi \times 10^7 \text{ s}$ $t_{\text{total-days}} = 365.24 \text{ days}$	$\text{PE} = \frac{\text{uncertainty}}{\text{value in question}} 100\%$	percent error = ?

• Convert Units

[2nd – Formulate Data] 50% of the credit on exams.

Let's first get everything in the correct units.

$$\left(\frac{365.24 \text{ days}}{1}\right)\left(\frac{24 \text{ hrs}}{1 \text{ days}}\right)\left(\frac{60 \text{ min}}{1 \text{ hrs}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 3.1557 \times 10^7 \text{ s}$$

$$\pi \times 10^7 \text{ s} = 3.1416 \times 10^7 \text{ s}$$

Now, to calculate the percent error, we can compare the two.

NOTE The "Percent Error" is also referred to as "Fractional Error."

Take the accepted value and subtract the approximate value to find the uncertainty.

$$|\pi \times 10^7 \text{ s} - 3.1557 \times 10^7 \text{ s}| = \pm 1.4 \times 10^5 \text{ s}$$

Take the uncertainty, divide the approximate value by the uncertainty and then multiple by 100%.

$$\frac{1.4 \times 10^5 \text{ s}}{3.1416 \times 10^7 \text{ s}}(100\%) = 0.4456\%$$

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[3rd – State the Answer] 0% of the credit on exams (but no credit without it).

Thus, to the correct significant figures, the percent error, is 0.45%.

Summing Vectors

For summation in vector notation(s), we will begin with 3D as most physics and engineering students will remain in three-dimensional space due to the structure of your physical body and mind and application of it in this realm. Mathematicians will generally branch to nth dimension indexing for theory and further applications of mathematics in higher order dimensional space related to theoretical physics and computer programming.

To begin, we will define notation. For this book—specifically, you can assume that a bold-non-italicized-lower-case letter is a column vector—that is,

$$\mathbf{x} = (x_1, x_2, x_3, ..) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = [x_1 \quad x_2 \quad x_3 \quad \cdots]^T, \qquad \mathbf{x} \in \mathbb{R}^n.$$

Now, let us define, "Unit Vectors." For the time being, we will remain in $\mathbb{R}^3$ to prepare for standard Tensor Analysis(Calculus).

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now, let $\mathbf{x} = (x_1, x_2, x_3)$. Following standard(universal) axiomatic properties for vector space:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \sum_{i=1}^{3} x_i\mathbf{e}_i \equiv x_i\mathbf{e}_i \quad \text{[Einstein Summation Convention]}.$$

It can then easily be shown that,

$$= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + \cdots + x_n\mathbf{e}_n = \sum_{i=1}^{n} x_i\mathbf{e}_i \equiv x_i\mathbf{e}_i$$

Column Vector Notation

$$\mathbf{v} = v_i\mathbf{e}_i, \qquad \mathbf{v} \in \mathbb{R}^n.$$

Engineers and Physicists (might find themselves using the following)

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\langle 1,0,0\rangle + v_2\langle 0,1,0\rangle + v_3\langle 0,0,1\rangle = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 \equiv v_i\hat{e}_i, \qquad \vec{v} \in 3\text{D}.$$

Column vs Row Vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_i\mathbf{e}_i, \qquad \mathbf{x}^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T = (x_1 \quad x_2 \quad x_3) = [x_i\mathbf{e}_i]^T = x_i[\mathbf{e}_i]^T = x_i\mathbf{e}^i.$$

Row Vector Notation (Transposed Column Vector)

$$\mathbf{v}^T = v_i\mathbf{e}^i, \qquad \mathbf{v} \in \mathbb{R}^n.$$

The Dot Product (for dot product & cross product, we will always be in 3D)

NOTE When we are using "hat notation," we are specifically in 3D or $\mathbb{R}^3$ in this book. When we use bold-non-italicized, we are in nth dimensional space or $\mathbb{R}^n$.

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = (u_1)(v_1) + (u_2)(v_2) + (u_3)(v_3)$$

$$= u_1v_1 + u_2v_2 + u_3v_3 = \sum_{i=1}^{3} u_iv_i.$$

The above is a summation but it is not good enough for what we need. Let us go deeper:

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = [\langle u_1, u_2, u_3 \rangle] \cdot [\langle v_1, v_2, v_3 \rangle]$$

$$= [u_1\langle 1,0,0\rangle + u_2\langle 0,1,0\rangle + u_3\langle 0,0,1\rangle] \cdot [v_1\langle 1,0,0\rangle + v_2\langle 0,1,0\rangle + v_3\langle 0,0,1\rangle]$$

$$= [u_1\langle 1,0,0\rangle + u_2\langle 0,1,0\rangle + u_3\langle 0,0,1\rangle] \cdot [v_1\langle 1,0,0\rangle + v_2\langle 0,1,0\rangle + v_3\langle 0,0,1\rangle]$$

$$= u_1\langle 1,0,0\rangle \cdot v_1\langle 1,0,0\rangle + u_2\langle 0,1,0\rangle \cdot v_2\langle 0,1,0\rangle + u_3\langle 0,0,1\rangle \cdot v_3\langle 0,1,0\rangle$$

$$= u_1\hat{e}_1 \cdot v_1\hat{e}_1 + u_2\hat{e}_2 \cdot v_2\hat{e}_2 + u_3\hat{e}_3 \cdot v_3\hat{e}_3$$

$$= u_1v_1\hat{e}_1 \cdot \hat{e}_1 + u_2v_2\hat{e}_2 \cdot \hat{e}_2 + u_3v_3\hat{e}_3 \cdot \hat{e}_3$$

NOTE It should be obvious the $\hat{e}_i \cdot \hat{e}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

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$$= u_1v_1(1) + u_2v_2(1) + u_3v_3(1) = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{i=1}^{3} u_iv_i.$$

Or,

$$= u_1v_1\hat{e}_1 \cdot \hat{e}_1 + u_2v_2\hat{e}_2 \cdot \hat{e}_2 + u_3v_3\hat{e}_3 \cdot \hat{e}_3 = \sum_{i=1}^{3} u_iv_i\, \hat{e}_i \cdot \hat{e}_i.$$

This is still not good enough....

[RP] Before going forward, can you get the above dot product into a double summation? We will do this in few pages. Try to think for yourself and show a double summation with i, j rather than just using i.

The Inner Product

The Inner Product is basically the same thing. The cross product only works in $\mathbb{R}^3$ and $\mathbb{R}^7$ that we know of right now, Earth Year 2026.

Now for the inner product notation, will be in nth dimensions using bold-non-italicized.

NOTE To reiterate: If we have hats, we are in 3D; if no hats, nth dimensions.

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Execute the "Inner Product" for nth dimensional space to find the indexed notation.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix}^T \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} = (u_1 \quad u_2 \quad \cdots)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} = u_1v_1 + u_2v_2 + \cdots = \sum_{i=1}^{n} u_iv_i$$

Not good enough!

$$= [u_1(1 \quad 0 \quad \cdots) + u_2(0 \quad 1 \quad \cdots) + \cdots]\left[v_1\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} + v_2\begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix} + \cdots\right]$$

$$= u_1(1 \quad 0 \quad \cdots)v_1\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} + u_2(0 \quad 1 \quad \cdots)v_2\begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix} + \cdots$$

$$= u_1[\mathbf{e}_1]^Tv_1\mathbf{e}_1 + u_2[\mathbf{e}_2]^Tv_2\mathbf{e}_2 + \cdots$$

$$= u_1v_1[\mathbf{e}_1]^T\mathbf{e}_1 + u_2v_2[\mathbf{e}_2]^T\mathbf{e}_2 + \cdots$$

$$= u_1v_1\mathbf{e}^1\mathbf{e}_1 + u_2v_2\mathbf{e}^1\mathbf{e}_2 + \cdots.$$

Clearly,

$$\mathbf{e}^i\mathbf{e}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

The Kronecker delta

From ChatGPT Sourcing [1]

The **Kronecker delta** δ_{ij} is named in honor of the German mathematician **Leopold Kronecker (1823–1891)** because of his influential contributions to algebra, number theory, and the foundations of mathematics. Although the symbol itself was introduced after many of his original works, the mathematical community adopted the name Kronecker delta to recognize his lasting impact on discrete mathematics and tensor notation. Today, the Kronecker delta is a standard mathematical symbol used throughout linear algebra, tensor analysis, engineering, and physics.

[RP] For students of PLEM Academy(University), please read the document found and cross reference their references to verify the previous paragraph is indeed accurate.

.edu Source Used:

[1] Texas A&M University, Lectures on Applied Mathematics (Section 1.1)

https://oaktrust.library.tamu.edu/server/api/core/bitstreams/43a7c801-5a35-41b6-a486-b8a8c3e1b16d/content

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From unit vector operations, it is painfully obvious—usually referred to as "clearly," that the product of unit vectors is the same thing as the Kronecker delta.

$$\delta_{ij} \equiv \mathbf{e}_i\mathbf{e}_j \equiv \mathbf{e}^i\mathbf{e}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Now, in the inner product and dot product, we did not use *j*; we only used *i*. I gather the reason why the lower-case Greek letter with indices is used is to reduce writing the continued dot product notation.

$$\therefore \quad \vec{v} = u_1v_1\hat{e}_1 \cdot \hat{e}_1 + u_2v_2\hat{e}_2 \cdot \hat{e}_2 + u_3v_3\hat{e}_3 \cdot \hat{e}_3$$

$$= u_1v_1\delta_{11} + u_2v_2\delta_{22} + u_3v_3\delta_{33}$$

We know,

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

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Thus, $3 \times 3 = 9$. For a double summation in 3D, we need 9 objects. Therefore,

$$= u_1v_1\delta_{11} + 0 + 0 + 0 + u_2v_2\delta_{22} + 0 + 0 + 0 + u_3v_3\delta_{33}$$

$$= u_1v_1\delta_{11} + u_1v_2\delta_{12} + u_1v_3\delta_{13}$$

$$+ u_2v_1\delta_{21} + u_2v_2\delta_{22} + u_2v_3\delta_{23}$$

$$+ u_3v_1\delta_{31} + u_3v_2\delta_{33} + u_3v_3\delta_{33}$$

Before continuing, notice the "matrix connection" of the Kronecker delta. I.e. (such that or that is),

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \Delta_3.$$

$$= u_1v_1\delta_{11} + u_1v_2\delta_{12} + u_1v_3\delta_{13}$$

$$+ u_2v_1\delta_{21} + u_2v_2\delta_{22} + u_2v_3\delta_{23}$$

$$+ u_3v_1\delta_{31} + u_3v_2\delta_{33} + u_3v_3\delta_{33}.$$

$$\therefore \quad \vec{u} \cdot \vec{v} = \sum_{i=1}^{3}\sum_{j=1}^{3} u_iv_j\delta_{ij} \equiv u_iv_j\delta_{ij}.$$

Now, this is much more versatile for index mathematics as it will now mix with higher order but is still not good enough(maybe).

Dot Product in 3D

$$\vec{u} \cdot \vec{v} = u_iv_j\delta_{ij}, \qquad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Inner Product in $\mathbb{R}^n$

$$\langle \mathbf{u}, \mathbf{v} \rangle u_i v_j \delta_i^j, \qquad \delta_i^j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

We read in alphabetical order from top to bottom.

Summing Polynomials

We will start with a standard polynomial in nth dimensions.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n = \sum_{m=0}^{n} a_m x^m$$

$$\sum_{m=0}^{n} a_m x^m$$

We have to be careful because x is a variable in a 2D coordinate system and is not matching with our previous notations. Hence, the use of m instead of i, j, k. Can we use this? Let us look at it from a different angle—that is, in vector form.

Let,

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

We see that it is an inner product.

$$= [a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_n] \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ \vdots \\ x^3 \end{bmatrix}.$$

Or

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n = f(x) = x^0 a_0 + x^1 a_1 + x^2 a_2 + \cdots + x^n a_n$$

$$= [x^0 \quad x^1 \quad x^2 \quad \cdots \quad x^n] \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [1 \quad x \quad x^2 \quad \cdots \quad x^n] \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

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$$= [1 \quad x \quad x^2 \quad \cdots \quad x^n] \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = (1 \times n)(n \times n)(n \times 1) = (1 \times 1) = \text{scalar}.$$

Or let the $\mathcal{P}_n = \{1, x, x^2, \dots, x^n\}$ and shorthand notate as follows.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = [1 \quad x \quad x^2 \quad \cdots \quad x^n] \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \equiv \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{\mathcal{P}_n} = \mathbf{f}.$$

$$\therefore \quad \mathbf{f} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{\mathcal{P}_n} .$$

Now, we have a vector that we can use in various polynomial space operations which can be put into summation and universal linear algebra techniques apply.

[RP] Show that $\mathbf{f} = [f_i \mathbf{e}_i]_{\mathcal{P}_n}$.

Introduction to Notations in Linear Algebra and ODE

Standard Derivative Notation(s)

Newton Fly Swat (Engineers)	Leibniz Operator (Universal)	Prime (Newton/Universal)
$\dot{x} = x'(t) = \frac{dx}{dt}$	$\frac{dy}{dx} = f'(x)$	$y' = \frac{dy}{dx}$
Dot notation is always with respect to time.		

To get an idea of how linear algebra and differential equations will come together, we start with some basic relations between physics, math, engineering fused with differential equations and linear algebra techniques.

Let's start with some calculus for constant acceleration. Let acceleration, a be constant and x represent distance. Then,

$$\ddot{x} = x''(t) = \frac{d^2x}{dt^2} = \frac{d}{dt}\frac{dx}{dt} = \frac{d}{dt}v = v'(t) = \frac{dv}{dt} = a(t) \quad \Rightarrow \quad \frac{dv}{dt} = a.$$

This previous equation is an English sentence—that is, "The second derivative of position with respect to time is equal to the first derivative of velocity with respect to time, which is equal to the derivative of velocity with respect to time, which is the acceleration as a function of time. Therefore, the derivative of velocity with respect to time equals acceleration." Make sure to punctuate.

Now, we will integrate to derive kinematic equation of motion on x-axis.

$$\frac{dv_x}{dt} = a_x \;\Rightarrow\; dv_x = a_x\,dt \;\Rightarrow\; \int dv_x = \int a_x\,dt \;\Rightarrow\; v_x + c_1 = a_x t + c_1$$

$$\Rightarrow\; v_x = a_x t + (c_2 - c_1) \;\Rightarrow\; v_x = a_x t + c_3 \;\Rightarrow\; v_x = a_x t + c_0 \;\equiv\; v_x = a_x t + v_{x,0}$$

$$\Rightarrow \frac{dx}{dt} = a_x t + v_{x,0} \;\Rightarrow\; dx = \left(a_x t + v_{x,0}\right)dt \;\Rightarrow\; dx = \left(a_x t + v_{x,0}\right)dt$$

$$\Rightarrow \int dx = \int a_x t + v_{x,0}\,dt \;\Rightarrow\; x = \frac{1}{2}a_x t^2 + v_{x,0}t + x_0.$$

We have just used basic calculus techniques to derive the one-dimensional kinematic equation of motion,

$$\Delta x = v_{x,0}t + \frac{1}{2}a_x t^2.$$

Now, what are we looking at in terms of differential equations?

$$\Delta x = v_{x,0} + \frac{1}{2}a_x t^2 \;\Rightarrow\; \frac{1}{2}a_x t^2 + v_{x,0}t + x_0 = x.$$

We know

$$a_x = \frac{d^2x}{dt^2} = x''(t) = \ddot{x}, \qquad v_x = \frac{dx}{dt} = x'(t) = \dot{x}, \qquad x(t) = x.$$

Usually Physics/Math	Usually Engineering	Usually Math/Physics
$\frac{1}{2}t^2x'' + tx' - x = x_0$	$\frac{1}{2}t^2\ddot{x} + t\dot{x} - x = x_0$	$\frac{1}{2}t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = x_0$

What we have here is a "Cauchy-Euler" formed (for the homogeneous side of the equation) first order differential equation non-homogeneous. (It is pronounced home – ah – gene – E – us. There is no 'z' so it is not potato-po tah toe. Homogenization is a term for separating fat droplets from milk.)

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For the homogeneous side of the ODE, we set the dependent operations equation to zero. That means, any of the objects that do not contain a dependent variable, are solve for the particular equation—that is,

$$\frac{1}{2}t^2x'' + tx' - x = x_0 \;\Rightarrow\; \frac{1}{2}t^2x_h'' + tx_h' - x_h = 0, \qquad \frac{1}{2}t^2x_p'' + tx_p' - x_p = x_0.$$

Cauchy-Euler Method for second-order.

$$at^2x'' + btx' + cx = 0, \qquad t > 0.$$

Let $x = t^r$. Then, (taking a lucky guess)

$$x = t^r \;\Rightarrow\; x' = rt^{r-1} \;\Rightarrow\; x'' = r(r-1)t^{r-2} \text{ [Power Rule].}$$

Then,

$$\frac{1}{2}t^2x'' + tx' - x = \frac{1}{2}t^2[r(r-1)t^{r-2}] + t[rt^{r-1}] - [t^r] = 0$$

$$\Rightarrow \frac{1}{2}r(r-1)t^2t^{r-2} + rtt^{r-1} - t^r = \frac{1}{2}r(r-1)t^{r-2+2} + rt^{r-1+1} - t^r$$

$$= \frac{1}{2}r(r-1)t^r + rt^r - t^r = t^r\left[\frac{1}{2}r(r-1) + r - 1\right]$$

$$= t^r\left[\frac{1}{2}r^2 - \frac{1}{2}r + r - 1\right] = t^r\left[\frac{1}{2}r^2 + \frac{1}{2}r - 1\right] = 0$$

$$\Rightarrow\; t^r > 0, \qquad r^2 + r - 2 = (r-1)(r+2) = 0 \;\Rightarrow\; r = 1 \;\vee \text{(or)}\; r = -2.$$

There are three general solutions to the Cauchy-Euler Method. Since we have two real solutions, the answer is quite basic: $x = c_1t^{r_1} + c_2t^{r_2}$.

Therefore,

$$x(t) = c_1t^1 + c_2t^{-2} = c_1t + \frac{c_2}{t^2}.$$

[WOLF](Wolfram Alpha) I used Wolfram Alpha(NOT A.I.) to check my answer—not solution.

We solved the homogeneous side of the ODE. For simplicity, let's set $x_0 = 0$. Thus, the equation is solved. What about the constants?

[RP] For c_1, c_2, how do they relate to the physics equation—that is, the kinematic equation we derived. HINT Initial Values. Also, are there any discrepancies ... does it make sense? Can we reverse this?

[FC] Can we show the answer is satisfactory for our kinematic equation? This is where we can verify our work.

FULL CIRCLE

Clearly, if

$$x = c_1 t + c_2 t^{-2}, \qquad x' = c_1 - 2c_2 t^{-3}, \qquad x'' = 6c_2 t^{-4}.$$

Then,

$$\frac{1}{2} a_x t^2 + v_{x,0} t - x = \frac{1}{2}[6c_2 t^{-4}]t^2 + [c_1 - 2c_2 t^{-3}]t - [c_1 t + c_2 t^{-2}] = 0$$

$$\Rightarrow \quad 3c_2 t^{-4} t^2 + c_1 - 2c_2 t^{-3} t - c_1 t - c_2 t^{-2} = 0$$

$$\Rightarrow \quad 3c_2 t^{-4+2} + c_1 - 2c_2 t^{-3+1} - c_1 t - c_2 t^{-2} = 0$$

$$\Rightarrow \quad 3c_2 t^{-2} + c_1 - 2c_2 t^{-2} - c_1 t - c_2 t^{-2} = 0$$

$$\Rightarrow \quad c_1 - c_1 t = 0.$$

TYPO ↑ We made some mistake somewhere. If the t isn't there, then it is balanced.

I will now plug the original ODE we derived into Wolfram Alpha to see what there answer is vs our answer.

I copied and pasted solve 1/2 t^2 x'' + tx' - x=0 the above equation into Wolfram Alpha and deleted the caret ^ between the variable and the prime. We have the same answer.



Image from WolframAlpha.com for educational purposes

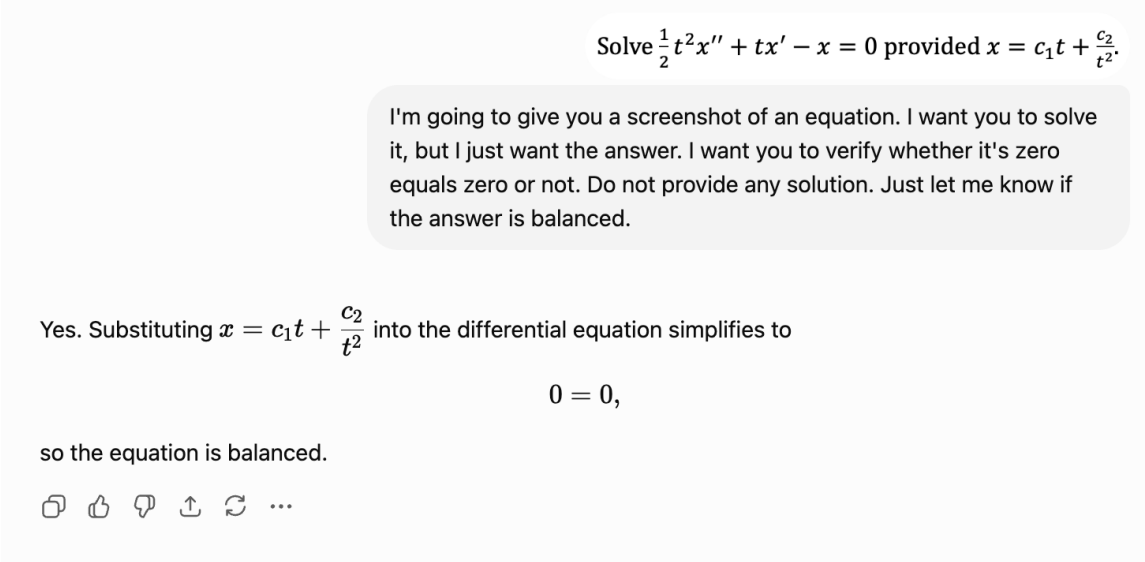
Now, to check my work again, I am going to request Wolfram Alpha to solve the ODE with the answer found.

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Solve $\frac{1}{2}t^2x'' + tx' - x = 0$, $x = c_1t + \frac{c_2}{t^2}$ ... Wolfram is not working out. I will use ChatGPT but I will request that it only verifies if it is $0 = 0$ and to specify not to provide any solution. I will use a screen shot of

Solve $\frac{1}{2}t^2x'' + tx' - x = 0$ provided $x = c_1t + \frac{c_2}{t^2}$.

The following image is from ChatGPT



[RP] Clearly, the answer is correct. Can you find the typo? I will address it in the follow up RECAP lesson.
Now, the next thing I want to look at, is the constant coefficient ODE. $ay'' + by' + cy = 0$.

For two real solutions, $y = c_1e^{m_1x} + c_2e^{m_2x}$.

There is a standard solution to this which is nearly identical to the Cauchy-Euler method. Only, we would use $y = e^{mx}$. We will come back to these techniques later on in the book. Now, we wish to look at how an ODE at the most basic level can become a system of differential equations for which linear algebra techniques are used to solve the ODE. For example, let

$$y'' + y' - 2y = 0.$$

Then,

$$y = e^{mx}, \qquad y' = me^{mx}, \qquad y'' = m^2e^{mx}$$

$$\Rightarrow \; y'' + y' - 2y = [m^2e^{mx}] + [me^{mx}] - 2[e^{mx}] = 0$$

$$\Rightarrow \; m^2e^{mx} + me^{mx} - 2e^{mx} = e^{mx}[m^2 + m - 2] = e^{mx}(m-1)(m+2) = 0$$

$$\therefore \; e^{mx} > 0, \qquad m = -2 \;\text{ or }\; m = 1.$$

Therefore,

$$y = c_1 e^{-2x} + c_2 e^{x}.$$

Basic Linear Algebra and Differential Equation Relation

In some books, $m^2 + m - 2 = 0$, is called the auxiliary equation. It is also the "Characteristic Polynomial" from the determinant of a matrix and $m = \lambda$, the eigenvalue(s).

As an engineer, you will examine the eigenvalues from the previous equations in relation to vector graphs and swirling stuff.

We will now write the second-order-ODE as a system of equations and use linear algebr to find the eigenvalues.

Let $y = x_1$ and $y' = x_2$. Then

$$y'' + y' - 2y = 0 \;\Rightarrow\; \begin{matrix} y = x_1 \\ y' = x_2 \end{matrix} \;\Rightarrow\; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} x_2 \\ 2y - y' \end{pmatrix} = \begin{pmatrix} x_2 \\ 2x_1 - x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\Rightarrow\; \frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \mathbf{x} \;\equiv\; X' = AX.$$

There are a few different ways to solve this, we could use the summation

For simplicity, let us examine a common scalar integra, $x' = ax$ and solve it.

$$\frac{dx}{dt} = ax \;\Rightarrow\; \frac{1}{x}\, dx = a\, dt \;\Rightarrow\; \ln|x| = at + c_0 \;\Rightarrow\; x = e^{at + c_0} = e^{at} e^{c_0} = c_1 e^{at}.$$

Thus,

$$x' = ax \quad \Rightarrow \quad x(t) = Ce^{at}.$$

Similar with a matrix, assume that $\mathbf{x}' = A\mathbf{x}$ has the answer, $\mathbf{x} = Ce^{At}$.

$$\exp\{at\} = e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} \quad \Rightarrow \quad \exp\{At\} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = e^{At}.$$

Depending on your major and or textbook and or level of math, you may use a pre-derived formula, or you may use the rigorous diagonalization method with the summation above. I will do a quick method.

[RP] Solve the system of equations with the summation and diagonalization.

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[RP] From the equation, $\mathbf{x}' = A\mathbf{x}$ for $A_{2\times 2}$—using linear algebra techniques (from any textbook)—show that $\mathbf{x} = Ce^{At}$ is the correct answer.

I will use a basic technique for system of ODE with a 2×2 matrix and a simple formula.

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \Rightarrow \det(A - I\lambda) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{vmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)(-1-\lambda) - (1)(2) = \lambda + \lambda^2 - 2 = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda_1 = -2, \qquad \lambda_2 = 1.$$

We found the "Eigenvalues." We have two real Eigenvalues. The solution for a system of second order homogenous differential equations with constant matrix is

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}.$$

We now must find the Eigenvectors.

$$\lambda_1: \quad \left[\begin{array}{cc|c} -(-2) & 1 & 0 \\ 2 & -1-(-2) & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\lambda_2: \quad \left[\begin{array}{cc|c} -(1) & 1 & 0 \\ 2 & -1-(1) & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Using the "Ghost Matrix" technique, we see

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus,

$$\mathbf{x} = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}.$$

[RP] Can you show that the matrix solution satisfies the original equation using the full circle of math? Also, can you show the diagonalization method can be rewritten as the vector solution in the previous line?

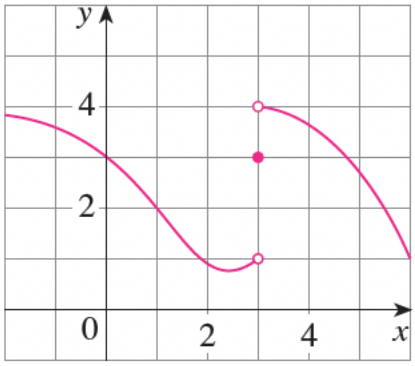
***[RP]** After reviewing the Constant Coefficient Method, do you think you find a way to put the Cauchy-Euler Method into a system of differential equations?

2.2: The Limit of a Function

QUESTION 2.2.5 [title of an essay]

For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, **explain why**.

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3^-} f(x)$ (c) $\lim_{x\to 3^+} f(x)$
(d) $\lim_{x\to 3} f(x)$ (e) $f(3)$



ANSWER

(a) 2 (b) 1 (c) 4 (d) Does not exist (e) 3

SOLUTION

$\mathcal{F}_1$ [Formulae (introduction of an essay and citation/reference)] 50% of EXAM Credit

$$\lim_{x\to a} f(x) = f(a) = L \text{ if } f(x) \text{ is left/right continuous,}$$

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L.$$

$\mathcal{F}_2$ [Formulate (body of the essay)] 50% of EXAM Credit

[a]

For (a), we are looking at a continuous left and right position on the graph. If it is continuous, then $\lim_{x\to a} f(x) = f(a) = L$.

$$\lim_{x\to 1} f(x) = f(1) = 2$$

[b]

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For (b), there is a hole in the graph and at the hole in the graph, the left-hand limit exists.

NOTE The function (if there were not a solid dot), would be "undefined" whereas the limit exists.

$$\lim_{x\to 3^-} f(x) = f(3) = 1.$$

[c]

At this point, we already referenced all the correct formulae, definition, and theorem. So, we can just crunch it out.

$$\lim_{x\to 3^+} f(x) = 4$$

[d]

Because $L^- = 1 \neq 4 = L^+$

$$\therefore \lim_{x\to 3} f(x) = \text{DNE}.$$

[e]

Because there is a solid dot, the function is "defined."

$$\therefore f(3) = 3.$$

$\mathcal{F}_3$ [Finalize (summarization of the essay)] 0% of EXAM Credit

(a) 2 (b) 1 (c) 4 (d) Does not exist (e) 3

EXAM SOLUTION METHOD

$$\lim_{x\to a} f(x) = L \Leftrightarrow L^- = L^+, \qquad \lim_{x\to a} f(x) = f(a) = L$$

For (a), $f(x)$ is continuous at $x = 1$.

$$\lim_{x\to 1} f(x) = f(1) = 2$$

For (b), there is a hole in the graph and at the hole in the graph, the left-hand limit exists.

$$\lim_{x\to 3^-} f(x) = f(3) = 1.$$

For (c),

$$\lim_{x\to 3^+} f(x) = 4$$

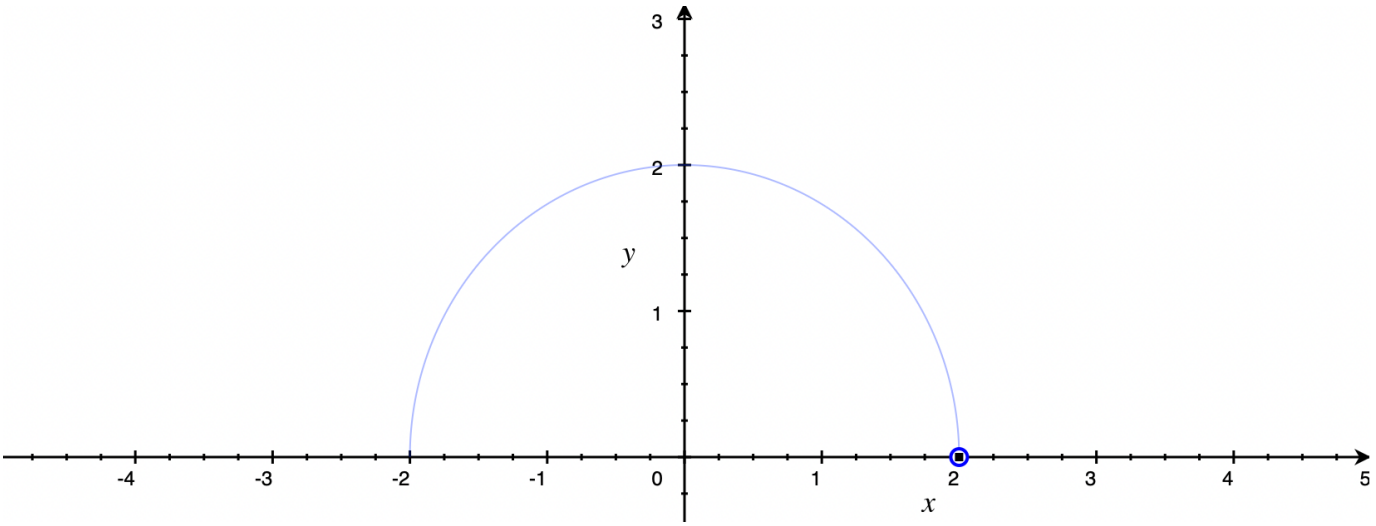
For (d), $L^- = 1 \neq 4 = L^+$

$$\therefore \lim_{x\to 3} f(x) = \text{DNE}.$$

For (e), because there is a solid dot, the function is "defined."

$$\therefore \ f(3) = 3.$$

Second Example



$$\lim_{x\to a} f(x) = f(a) = L$$

For another definition, as long as the function is continuous or defined at the point of reference as in the semi-circle above, by definition, the limit exists.

$$\lim_{x\to 2} \sqrt{4 - x^2} = \sqrt{4 - 2^2} = \sqrt{4 - 4} = \sqrt{0} = 0.$$

Thus

$$L = 0.$$

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2.1: The Tangent and Velocity Problems

For this particular potion of the textbook, all you need to do is thoroughly read it and work the example questions in the reading. It is intuitive and not necessary for me to work any questions.

DEFAULT Solution(Essay) Flow

QUESTION [Essay Title]

ANSWER

SOLUTION

$\mathcal{F}_1$ [Formulae (intro to essay)] 50% of Exam Credit (0% without the latter)

$\mathcal{F}_2$ [Formulate (body of essay)] 50% of Exam Credit (0% without the former)

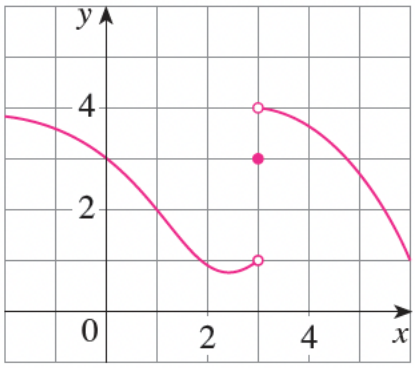
$\mathcal{F}_3$ [Finalize (summarization)] 0% of Exam Credit (No points without steps 1 or 2—in general)

2.2: The Limit of a Function

QUESTION 2.2.5 [title of an essay]

For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, **explain why**.

(f) $\lim_{x\to 1} f(x)$ (g) $\lim_{x\to 3^-} f(x)$ (h) $\lim_{x\to 3^+} f(x)$
(i) $\lim_{x\to 3} f(x)$ (j) $f(3)$



ANSWER

(f) 2 (g) 1 (h) 4 (i) Does not exist (j) 3

SOLUTION

$\mathcal{F}_1$ [Formulae (introduction of an essay and citation/reference)] 50% of EXAM Credit

$$\lim_{x\to a} f(x) = f(a) = L \text{ if } f(x) \text{ is left/right continuous,}$$

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L.$$

$\mathcal{F}_2$ [Formulate (body of the essay)] 50% of EXAM Credit

[a]

For (a), we are looking at a continuous left and right position on the graph. If it is continuous, then $\lim_{x\to a} f(x) = f(a) = L$.

$$\lim_{x\to 1} f(x) = f(1) = 2$$

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[b]

For (b), there is a hole in the graph and at the hole in the graph, the left-hand limit exists.

NOTE The function (if there were not a solid dot), would be "undefined" whereas the limit exists.

$$\lim_{x\to 3^-} f(x) = f(3) = 1.$$

[c]

At this point, we already referenced all the correct formulae, definition, and theorem. So, we can just crunch it out.

$$\lim_{x\to 3^+} f(x) = 4$$

[d]

Because $L^- = 1 \neq 4 = L^+$

$$\therefore \lim_{x\to 3} f(x) = \text{DNE}.$$

[e]

Because there is a solid dot, the function is "defined."

$$\therefore f(3) = 3.$$

$\mathcal{F}_3$ [Finalize (summarization of the essay)] 0% of EXAM Credit

(f) 2 (g) 1 (h) 4 (i) Does not exist (j) 3

EXAM SOLUTION METHOD [Ask your professor to grade this in office hours to make sure.]

$$\lim_{x\to a} f(x) = L \Leftrightarrow L^- = L^+, \qquad \lim_{x\to a} f(x) = f(a) = L$$

For (a), $f(x)$ is continuous at $x = 1$.

$$\lim_{x\to 1} f(x) = f(1) = 2$$

For (b), there is a hole in the graph and at the hole in the graph, the left-hand limit exists.

$$\lim_{x\to 3^-} f(x) = f(3) = 1.$$

For (c),

$$\lim_{x\to 3^+} f(x) = 4$$

For (d), $L^- = 1 \neq 4 = L^+$

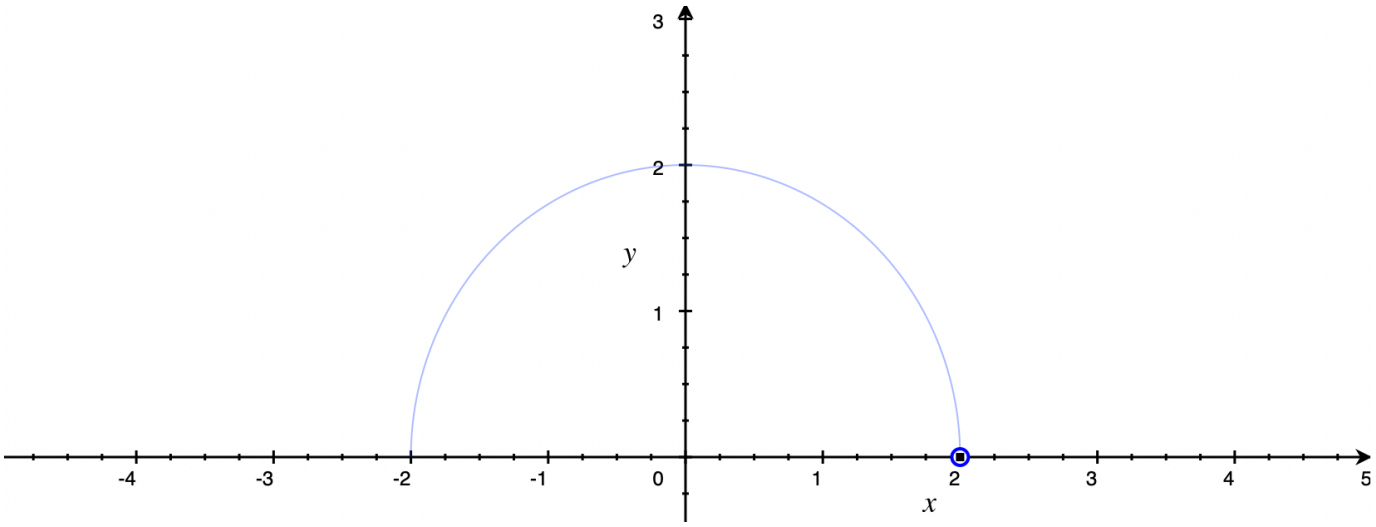
$$\therefore \lim_{x\to 3} f(x) = \text{DNE}.$$

For (e), because there is a solid dot, the function is "defined."

$$\therefore \ f(3) = 3.$$

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Second Example



$$\lim_{x \to a} f(x) = f(a) = L$$

For another definition, as long as the function is continuous or defined at the point of reference as in the semi-circle above, by definition, the limit exists.

$$\lim_{x \to 2} \sqrt{4 - x^2} = \sqrt{4 - 2^2} = \sqrt{4 - 4} = \sqrt{0} = 0.$$

Thus

$$L = 0.$$

Sketch the graph of the function and use it to determine the limit

---

QUESTION [Essay Title] 2.2.11

Sketch the graph of the function and use it to determine **the values of** a for which $\lim_{x\to a} f(x)$ exists.

$$f(x)=\begin{cases}1+x & \text{if } x<-1\\ x^2 & \text{if } -1\le x<1.\\ 2-x & \text{if } x\ge 1\end{cases}$$

ANSWER [Pg. A71]

$$\lim_{x\to a} f(x) \text{ exists for all } a \text{ except } a=-1.$$

SOLUTION

$\mathcal{F}_1$ [Formulae (intro to essay)] 50% of Exam Credit (0% without the latter)

$$f(x)=\begin{cases}1+x & \text{if } x<-1\\ x^2 & \text{if } -1\le x<1\\ 2-x & \text{if } x\ge 1\end{cases}$$

We may need to reference some definitions and theorems: We have a piecewise function; the definition for piecewise is different for defined functions. Meaning: there are more than one definition for limits.

STANDARD DEFINITION(s) of Limits

Two-Sided Limit

$$\lim_{x\to a} f(x)=L \iff L^-=L^+ \text{ [not necessarily defined]}$$

One-Sided or Defined at $x=a$

$$\lim_{x\to a} f(x)=f(a)=L \text{ [defined at } x=a]$$

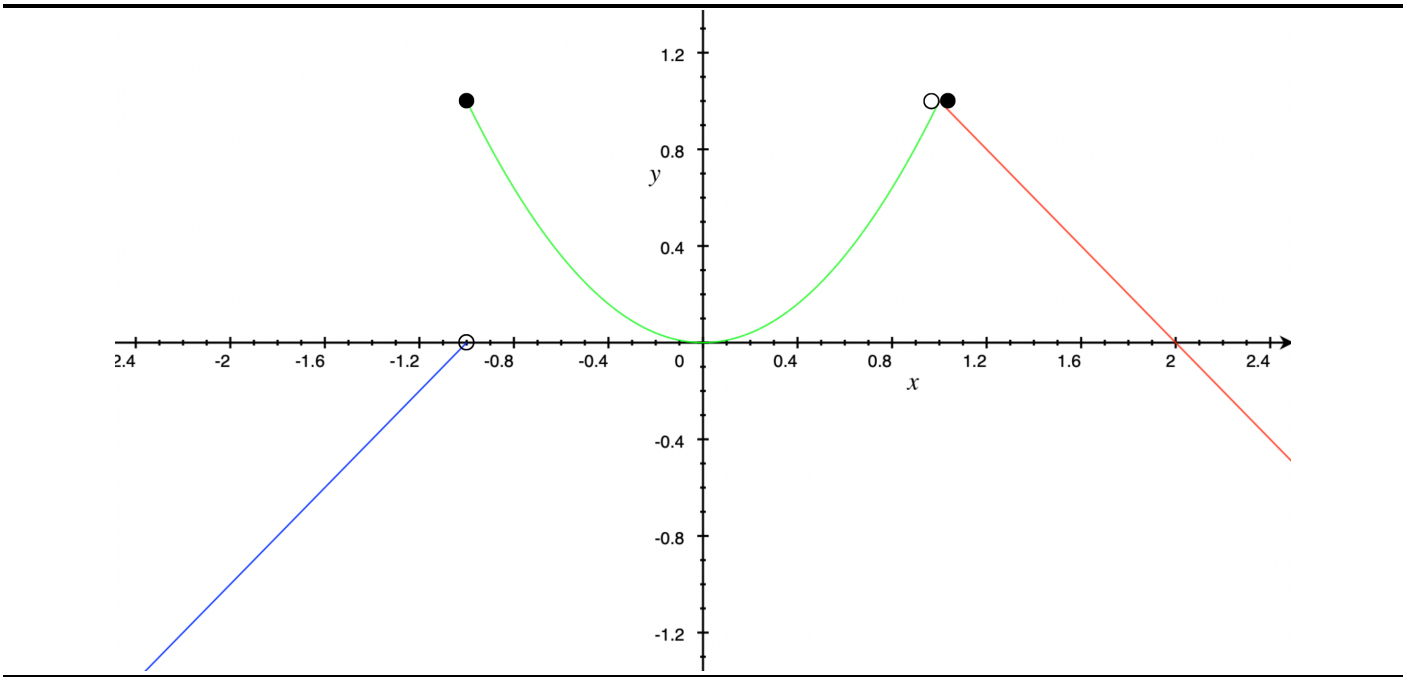
I will use the built in grapher for the MacBook under utilities.

$\mathcal{F}_2$ [Formulate (body of essay)] 50% of Exam Credit (0% without the former)

In the video lesson, we are graphing in MacBook grapher and using Canva to put holes in the graph and continuous dots.

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[TM (time management)] On an exam, the question did not specify to label the graph, so don't waste time doing that. If you lose points, you can argue the professor that the instructions did not say to do. Thus, it is not necessary.



[$y = 1 + x$] For this question, at $y = 1 + x$, the function is discontinuous at $x = -1$. Thus, it has no limit for a one-sided limit, and the right-hand side does not equal -1. There is no limit at $x = -1$.

There is a left-hand limit

$$\lim_{x\to -1^-} f(x) = 0, \qquad \lim_{x\to -1^+} f(x) = 1, \qquad f(-1) = \text{undefined},$$

$$\lim_{x\to -1} f(x) = \text{DNE} \quad \because L^- \neq L^+.$$

[$y = x^2$] [$y = 1 - x$]

Now, we see from the graph, at $x = 1$, $L^- = 1 = L^+$.

$$\therefore \lim_{x\to 1} f(x) = 1 = \text{EXISTS.} \qquad f(1) = 1 = \text{defined on } [1, \infty).$$

$\mathcal{F}_3$ [Finalize (summarization)] 0% of Exam Credit (No points without steps 1 or 2—in general)

Therefore, according to the **definition of a limit**, we see that the limit exists everywhere except when $x = -1$.

2.3: Calculating Limits Using the Limit Laws

Evaluating a limit by function

QUESTION [Essay Title] 2.3.1.c, 2.3.1.f

Given that

$$\lim_{x\to 2} f(x) = 4 \qquad \lim_{x\to 2} g(x) = -2 \qquad \lim_{x\to 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

$$\text{(c) } \lim_{x\to 2} \sqrt{f(x)}, \qquad \text{(d) } \lim_{x\to 2} \frac{g(x)h(x)}{f(x)}.$$

ANSWER

$$2, \qquad 0$$

SOLUTION

$\mathcal{F}_1$ [Formulae (intro to essay)] 50% of Exam Credit (0% without the latter)

LIMIT LAWS

Product Law

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} g(x) \cdot \lim_{x\to a} f(x)$$

Quotient Law

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}, \qquad \lim_{x\to a} g(x) \neq 0$$

Root Law

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}, \qquad n \in \mathbb{N} = \{1,2,3,\dots\}, \qquad \lim_{x\to a} f(x) > 0$$

$\mathcal{F}_2$ [Formulate (body of essay)] 50% of Exam Credit (0% without the former)

[c] Using the “Root Law”

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$$\lim_{x\to 2}\sqrt{f(x)}=\sqrt{\lim_{x\to 2}f(x)}=\sqrt{4}=\sqrt{2^2}=(2^2)^{\frac{1}{2}}=2^{2\left(\frac{1}{2}\right)}=2^{\frac{2}{2}}=2^1=2.$$

[d] Product Law and Quotient Law

$$\lim_{x\to 2}\frac{g(x)h(x)}{f(x)}=\frac{\lim_{x\to 2}g(x)h(x)}{\lim_{x\to 2}f(x)}=\frac{\lim_{x\to 2}g(x)\cdot\lim_{x\to 2}h(x)}{\lim_{x\to 2}f(x)}=\frac{(-2)(0)}{(4)}=\frac{0}{4}=0.$$

$\mathcal{F}_3$ [Finalize (summarization)] 0% of Exam Credit (No points without steps 1 or 2—in general)

Therefore, by the limit laws, Root, Product, and Quotient, we see

$$\lim_{x\to 2}\sqrt{f(x)}=2,\qquad \lim_{x\to 2}\frac{g(x)h(x)}{f(x)}=0.$$

EXAM METHOD

By the Root, Product, and Quotient Law:

$$\lim_{x\to 2}\sqrt{f(x)}=\sqrt{\lim_{x\to 2}f(x)}=\sqrt{4}=\sqrt{2^2}=(2^2)^{\frac{1}{2}}=2^{2\left(\frac{1}{2}\right)}=2^{\frac{2}{2}}=2^1=2,$$

$$\lim_{x\to 2}\frac{g(x)h(x)}{f(x)}=\frac{\lim_{x\to 2}g(x)h(x)}{\lim_{x\to 2}f(x)}=\frac{\lim_{x\to 2}g(x)\cdot\lim_{x\to 2}h(x)}{\lim_{x\to 2}f(x)}=\frac{(-2)(0)}{(4)}=\frac{0}{4}=0.$$

Introduction to Differentiation Notation

[SU (Slang Use)]

Prime Notation	Dot Notation (time)	Operator Notation	Partial Derivatives
$[y]'=\frac{dy}{dx}$	$\dot{y}=\frac{dy}{dt}=y'(t)$	$\mathcal{D}\{y\}=\frac{d}{dx}y=\frac{dy}{dx}$	$\frac{\partial}{\partial x}f=\frac{\partial f}{\partial x}=f_x$

For this book, Volume 6- Differential Calculus, we are strictly working from the Stewart Calculus Textbook. However, I will periodically mention [SU] for "Slang Use."

Prime Notation (Usually for Business Majors)

Use the *Power Rule* for Prime Notation. Let $y = \frac{3}{5}x^{\frac{5}{7}}$. By the power rule, $[x^n] = nx^{n-1}$.

$$n = \frac{5}{7}, \qquad \left[\frac{3}{5}x^{\frac{5}{7}}\right]' = \frac{3}{5}\left[x^{\frac{5}{7}}\right]' = \frac{3}{5}\left[\frac{5}{7}x^{\frac{5}{7}-1}\right] = \frac{3}{5}\left[\frac{5}{7}x^{\frac{5}{7}-\frac{7}{7}}\right] = \frac{3}{5}\left[\frac{5}{7}x^{-\frac{2}{7}}\right] = \frac{3}{7}x^{-\frac{2}{7}}.$$

Dot Notation for Time Derivatives (usually for engineers)

Use the *Product Rule* for dot notation (time derivatives).

$$[\dot{xy}] = x\dot{y} + y\dot{x} \equiv xy'(t) + yx'(t) = x\frac{dy}{dt} + y\frac{dx}{dt} \text{ [Implicit]}$$

Operator Notation (usually found in higher math)

Let $\mathcal{D}$ be the operator for which the derivative is taken(definition of $\mathcal{D}$).

Execute $\mathcal{D}\{\sin\cos x\}$

$$\mathcal{D}\{\sin\cos x\} = \cos\cos x\,\mathcal{D}\{\cos x\} = \cos\cos x\,(-\sin x) = -\sin x\cos\cos x\,.\text{ [Chain Rule]}$$

$$\frac{d}{dx}\sin(\cos x) = \frac{d}{dx}[\sin(\cos x)]^1 \text{ (by chain rule and power rule)}$$

When performing the chain rule, you perform power rule or trig identity and leave then parenthetical alone and multiple by the parenthetical's derivative. Since it is the power of 1, taking the power rule, would change it to a constant $\frac{d}{dx}x = (1)x^{1-1} = \frac{x}{x} = 1$. With trig, we must use the identity.

$$\text{Pointless Circle } \frac{d}{dx}[\sin(\cos x)]^1 = (1)[\sin(\cos x)]^{1-1}\frac{d}{dx}\sin(\cos x) = \frac{d}{dx}\sin(\cos x)$$

For trigonometry chain rule, if you have the power of 1, then you simply apply identity to the composing function:

$$\frac{d}{dx}[\sin(\cos x)]^1 = \frac{d}{dx}\sin(\cos x) = \cos\cos x\frac{d}{dx}\cos x = \cos\cos x\,(-\sin x) = -\sin x\cos\cos x.$$

[PL (Point Loss) TEST QUESTION] If you have a higher order exponent, then you do the power rule first. E.g.,

$$\frac{d}{dx}\sin^3\cos x = 3\sin^2\cos x\frac{d}{dx}\sin\cos x = 3\sin^2\cos x\,(-\sin x\cos\cos x)$$

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Using the chain rule—composition derivative formula,

$$\big([f \circ g](x)\big)' = \frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

$$\frac{d}{dx}\sin^3\cos x = \frac{d}{dx}[\sin\cos x]^3 = 3[\sin\cos x]^{3-1}\frac{d}{dx}\sin\cos x$$

$$f(x) = x^3, \qquad g(x) = \sin\cos x\,, \qquad f'(x) = 3x^2, \qquad g'(x) = -\sin x\cos\cos x.$$

$$\big([f \circ g](x)\big)' = \frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)g'(x) = 3x^2(-\sin x\cos\cos x)$$

$$= 3(\sin\cos x)^2(-\sin x\cos\cos x) = 3\sin^2\cos x\,(-\sin x\cos\cos x).$$

$$= -3\sin x\cos\cos x\sin^2\cos x.$$

During live session a students asked why I didn't use the product rule for

$$\frac{d}{dx}\sin(\cos x).$$

Because it is composition; not a product—that is (i.e.),

Let $f(x) = \sin x\,,\ g(x) = \cos x$. Then, $[f \circ g](x) = f\big(g(x)\big) = [\sin x \circ \cos x] = \sin\cos x$.

The composition functions derivative is the chain rule,

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)g'(x) = \frac{df}{dg}\frac{dg}{dx}, \qquad [u(v)]' = u'(v)v'.$$

Partial Derivates vs Single Variable Derivatives and Implicit Differentiation

Standard Derivative with Respect to x

$$\frac{d}{dx}x^n = nx^{n-1}\frac{d}{dx}x = nx^{n-1}\frac{dx}{dx} = nx^{n-1}(1) = nx^{n-1}$$

Implicit Derivative with Respect to x

$$\frac{d}{dx}y^n = ny^{n-1}\frac{d}{dx}y = ny^{n-1}\frac{dy}{dx} = ny^{n-1}y'$$

Implicit Differentiation with Respect to Time (product rule)

$$\frac{d}{dt}[x^n y^n] = x^n\frac{d}{dt}y^n + y^n\frac{d}{dt}x^n = x^n\left[ny^{n-1}\frac{dy}{dt}\right] + y^n\left[nx^{n-1}\frac{dx}{dt}\right]$$

$$= x^n[ny^{n-1}y'] + y^n[nx^{n-1}x'] = nx^n y^{n-1}y' + nx^{n-1}y^n x'$$

$$= nx^n y^{n-1}\dot{y} + nx^{n-1}y^n\dot{x}.$$

Partial Differentiation

$$\frac{d}{dx}kx^n = k\frac{d}{dx}x = k[nx^{n-1}] \text{ single variable}$$

$$\frac{\partial}{\partial t}[x^n y^n] = [x^n y^n]\frac{\partial}{\partial t}(1) = x^n y^n\frac{d}{dx}(1) = x^n y^n(0) = 0.$$

$$\frac{\partial}{\partial x}[x^n y^n] = y^n\frac{\partial}{\partial x}x^n = y^n\frac{d}{dx}x^n = y^n[nx^{n-1}] = nx^{n-1}y^n$$

$$\frac{\partial}{\partial y}[x^n y^n] = x^n\frac{\partial}{\partial y}y^n = x^n\frac{d}{dy}y^n = x^n[ny^{n-1}] = nx^n y^{n-1}$$

Notice the difference between implicit differentiation and partial.

$$\frac{d}{dx}[x^n y^n] = x^n\frac{d}{dx}y^n + y^n\frac{d}{dx}x^n = x^n\left[ny^{n-1}\frac{dy}{dx}\right] + y^n[nx^{n-1}]$$

$$= nx^n y^{n-1}y' + nx^{n-1}y^n.$$

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Beginner Intro: Single vs Double vs Triple Integral

For those of you who have taken a high school level calculus, you may already have been introduced to antiderivatives and integration. However, you are starting (like smart student) at Calculus 1—differential, even though you may have been able to test out.

[WP (Withdraw Prevention)] One of the biggest reasons students fail or withdraw Calculus 2—Integral, is because they tested out of Calculus 1—Differential due to their high school experience. It should not be allowed to do that. High school calculus is not even remotely close to college calculus. Start at Calculus 1—always.

To understand the arithmetic operation of a nth order integral, first consider a simple definite integral such as

$$\int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \left(\frac{1}{2}(1)^2\right) - \left(\frac{1}{2}(0)^2\right) = \frac{1}{2}.$$

Next, what is the antiderivative of 1 with respect to y?

$$\int dy = y + C$$

Consider undoing the definite integral leading to the answer 1 on the interval [0,1] with respect to y. E.g.,

$$1 = (1) - (0) = y|_0^1 = \int_0^1 dy.$$

Now, going back to $\int_0^1 x\,dx$,we see,

$$\int_0^1 x\,dx = \int_0^1 x(1)\,dx = \int_0^1 x\left[\int_0^1 dy\right]dx = \int_0^1 dy \int_0^1 x\,dx \text{ [not always possible]}$$

$$= \int_0^1 \int_0^1 x\,dxdy = \int_0^1 \int_0^1 \int_0^1 x\,dxdydz \ldots.$$

Conclusion- every time you integrate, you are integrating multiple integrals one way or another.

You are also solving differential equations in calculus 1/2—differential and integral. E.g., solve

$$y' = x \quad \Rightarrow \quad y = \frac{1}{2}x^2 + C. \text{ [Antiderivative with Power Rule]}$$

We can look at this with integration-

$$y' = x \quad \Rightarrow \quad \frac{dy}{dx} = x \quad \Rightarrow \quad dy = x\,dx \quad \Rightarrow \quad \int dy = \int x\,dx$$

$$y + c_1 = \frac{1}{2}x^2 + c_2 \quad \Rightarrow \quad y = \frac{1}{2}x^2 + c_2 - c_1 = \frac{1}{2}x^2 + c_3$$

$$\therefore \quad y = \frac{1}{2}x^2 + C. \text{ [Indefinite Integral with Power Rule]}$$

In ODE (Ordinary Differential Equations) We would a plethora of options to choose from. The previous method, in ODE, is called Separable Variable Method. In calculus, you are integrating. In ODE, you are using separable variable technique.

Now, I will use first order linear technique for ODE. Keep in mind, that there are probably an infinite number of ways to solve this with ODE's and finding a new method could a dissertation for PhD.

First-order-linear-nonhomogeneous ODE $y' = x \quad \Rightarrow \quad y' - (0)y = x$

There is straightforward formula for this method as follows.

$$y' + P(x)y = Q(x), \qquad I(x) = e^{\int P(x)\,dx}, \qquad y(x) = \frac{1}{I(x)}\left[\int Q(x)I(x)\,dx + C\right].$$

$$P(x) = 0, \qquad Q(x) = x, \qquad I(x) = e^{\int (0)\,dx} = e^{\text{constant}} = e^{c_1} = c_2.$$

$$y(x) = \frac{1}{I(x)}\left[\int Q(x)I(x)\,dx + C\right] = \frac{1}{c_2}\left[\int x(c_2)\,dx + C\right]$$

$$= c_2c_3\left[\int x\,dx + C\right] = c_4\left[\frac{1}{2}x^2 + C\right] = \frac{c_4}{2}x^2 + c_4C = c_5x^2 + C.$$

[RP (Research Position)] I used the first-order-linear method of ODE. But there is a problem because it is not the same answer as $\frac{1}{2}x^2 + C$. Now, why? This is some food for thought. I will give you a hint(maybe), did I use the formula correctly for $P(x) = 0$? Does $P(x) > 0$? Check any accredited textbook to verify. [CHALLENGE] Continue solving with more ODE Methods.

$$\text{[Correction]: } \frac{c_2}{c_2}\left[\int x\,dx + C\right] = (1)\left[\int x\,dx + C\right] = \frac{1}{2}x^2 + C.$$

Partial Derivative Case

$$\frac{\partial u}{\partial x} = u_x = x, \qquad u = u(x, y). \qquad \int \partial u = \int x\,\partial x \quad \Rightarrow \quad u = \frac{1}{2}x^2 + f(y).$$

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3.4 The Chain rule [TTS (test taking sample)]

QUESTION 3.4.45

$$y = \cos\sqrt{\sin(\tan \pi x)}$$

SOLUTION

NOTE For a [TTS], I omit the answer because we are going to solve it and leave mistakes in, if wrong, and use technology to verify the "answer"—not the solution. We never-ever look at solution unless it was written by the professor or the textbook.

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x) = \frac{df}{dg}\frac{dg}{dx} = f'(g)g' = u'(v)v'$$

Let $y = \cos\sqrt{\sin(\tan \pi x)}$. Find y'.

$$\frac{d}{dx}y = \frac{dy}{dx} = y' = \frac{d}{dx}\cos\sqrt{\sin(\tan \pi x)} = \frac{d}{dx}\cos\left(\sin^{\frac{1}{2}}(tan\, \pi x)\right) = \frac{d}{dx}\cos\left([\sin(\tan(\pi x)]^{\frac{1}{2}}\right)$$

$$= -\sin\left([\sin(\tan(\pi x)]^{\frac{1}{2}}\right)\frac{d}{dx}[\sin(\tan(\pi x)]^{\frac{1}{2}}$$

Power Rule with Chain Rule

$$\frac{d}{dx}x^n = nx^{n-1}\frac{d}{dx}x = nx^{n-1}\frac{dx}{dx} = nx^{n-1}(1) = nx^{n-1}$$

$$= -\sin\left[\sqrt{\sin(\tan \pi x)}\right]\left(\frac{1}{2}[\sin(\tan(\pi x)]^{\frac{1}{2}-1}\right)\frac{d}{dx}\sin(\tan(\pi x))$$

$$= -\sin\left[\sqrt{\sin(\tan \pi x)}\right]\left(\frac{1}{2}[\sin(\tan(\pi x)]^{\frac{1}{2}-\frac{2}{2}}\right)\cos(\tan(\pi x))\frac{d}{dx}\tan(\pi x)$$

$$= -\sin\left[\sqrt{\sin(\tan \pi x)}\right]\left(\frac{1}{2}[\sin(\tan(\pi x)]^{\frac{1}{2}-\frac{2}{2}}\right)\cos(\tan(\pi x))\,\pi\sec^2(\pi x) \text{ [final answer]}$$

NOTE On an exam, leave the previous line as the final answer unless the professor specifies to simplify.

$$= -\frac{\pi}{2}\sin\left[\sqrt{\sin(\tan \pi x)}\right]\left(\frac{1}{\sqrt{\sin(\tan \pi x)}}\right)\cos(\tan(\pi x))\sec^2(\pi x)$$

Now, I will check with Wolfram Alpha and the answer in the back of the book.

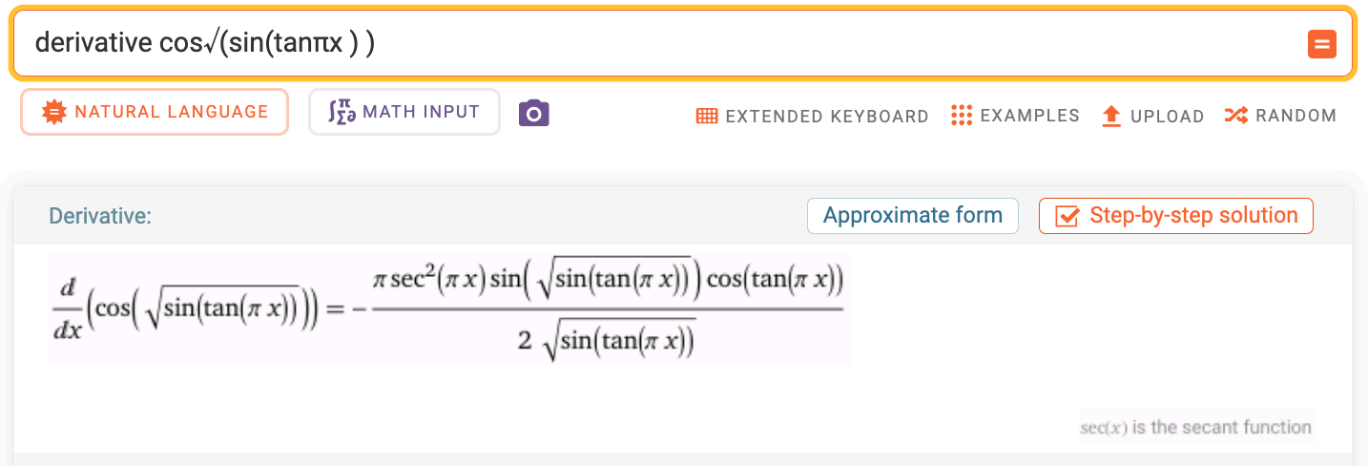


Image from Wolfram Alpha for educational purpose.

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45. $y' = \dfrac{-\pi \cos(\tan \pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$

Books answer (image for educational purposes).

ChatGPT Answer (we requestion explicitly not to provide a solution for that would be plagiarism—that is, we just asked for the answer.)

$$y' = -\frac{\pi \sec^2(\pi x)\cos(\tan(\pi x))\sin\left(\sqrt{\sin(\tan(\pi x))}\right)}{2\sqrt{\sin(\tan(\pi x))}}$$

Image from ChatGPT.com for education purposes.

Our answer is the same. Thus,

$$y' = -\frac{\pi}{2}\sin\left[\sqrt{\sin(\tan \pi x)}\right]\left(\frac{1}{\sqrt{\sin(\tan \pi x)}}\right)\cos(\tan(\pi x))\sec^2(\pi x)$$

$$= -\frac{\pi \cos(\tan(\pi x))\sec^2(\pi x)\sin\left[\sqrt{\sin(\tan \pi x)}\right]}{2\sqrt{\sin(\tan \pi x)}}.$$

Question Layout- Related Rates

QUESTION (Title of the Essay)

ANSWER (Summary of Essay)

SOLUTION (Intro, Body, and Summary of the Essay)

RELATED RATES Essay Writing

$\mathcal{F}_0$ [collect data]

Keywords:

Equations Needed	Given Data	Assemble a one variable function

$\mathcal{F}_1$ [Implicitly Differentiate w/Respect to Time]

$\mathcal{F}_2$ [Summarize]

For Related Rates, we are essentially executing algorithms for physics questions in theory–meaning non-physical—precise.

We look at physics problems differently than math problems and then there is a difference between calculus application and algebraic application of derived calculus formulae found in the standard physics textbooks. What I mean by that is, in your university physics course, you will use derived algebraic equations or formulas vs. assembling functions and using calculus techniques to identify rates of change, minimums, maximums from graphical relation and definition relation. Multiple worlds of physics and math problem solving.

QUESTION [6] 3.9.1

If V is the **volume of a cube** with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.

ANSWER [Page A80]

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

SOLUTION

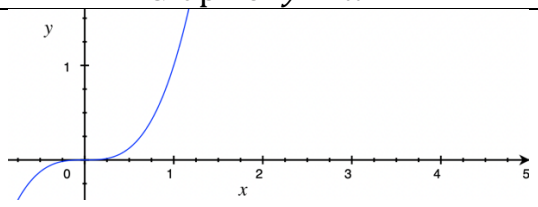
Keywords: volume of a cube

Equations Needed	Solve in One-Variable	One Variable Function
Cuboid (3D Calculus)	For a cuboid, we have	Thus,
$V = LWH = xyz$	$V(x, y, z) = xyz.$	$V(x) = x^3$
Cube $L = W = H = x$	We want this in one variable,	
$V = xxx = x^{1+1+1} = x^3$	$L = x \Rightarrow V = x^3.$	

[FR (Future Reference)] You will be asked the same question for cuboids and other 3D objects using techniques in 3D calculus with partial derivatives and something called Lagrangian Multipliers.

For Related Rates, once you have the single variable function, you simply implicitly differentiate with respect to time. Thus,

$$\frac{d}{dt}[V(x) = x^3] = \frac{dV}{dt} = \frac{d}{dt}x^3 = \frac{d}{dt}[x]^3 = 3[x]^{3-1}\frac{d}{dt}x = 3x^2\frac{dx}{dt}.$$

	Graph of $y = x^3$.
ANSWER Therefore, the rate of change is $V'(t) = 3x^2 \frac{dx}{dt}.$	

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Spherical Ball

QUESTION [1] 3.9.7 (Title of the Essay)

The radius of a **spherical ball** is increasing at a rate of 2 cm/min. At what rate is the **surface area** of the ball increasing when the radius is 8 cm?

ANSWER [Page A80] (Summary of Essay)

$$128\pi \ \frac{\text{cm}^2}{\text{min}}$$

SOLUTION (Intro, Body, and Summary of the Essay)

RELATED RATES Essay Writing $\mathcal{F}_0$ [collect data] Keywords: Sphere, Surface Area

Equations Needed	Values Given	Assemble a one variable function
$V_{\text{sphere}} = \frac{4}{3}\pi r^3$ From the reference page of Edition 7 of the Stewart Series: $SA_{\text{sphere}} = 4\pi r^2$	$\frac{dr}{dt} = 2 \ \frac{\text{cm}}{\text{min}}$ $r = 8 \text{ cm}$ $\frac{dSA}{dt} = ?$	$SA(r) = 4\pi r^2$

[RP] The derivative of the volume with respect to the radius, is equal to the surface area. Do you think there is a relation? $\frac{d}{dr}V = \frac{d}{dr}\frac{4}{3}\pi r^3 = \frac{4}{3}(3\pi r^{3-1}) = 4\pi r^2 = SA$.

$\mathcal{F}_1$ [Implicitly Differentiate w/Respect to Time] We are looking for the rate of change of the surface area. Thus, we must differentiate with respect to time, the equation (in oner variable) that represents the surface area.

$$\frac{d}{dt}[SA(r) = 4\pi r^2] \ \Rightarrow \ \frac{dSA}{dt} = 4\pi \frac{d}{dt} r^2 = 8\pi r \frac{dr}{dt}$$

$$SA'(t) = 8\pi r \frac{dr}{dt} \ \Rightarrow \ \frac{dSA}{dt} = 8\pi[8 \text{ cm}]\left[2 \ \frac{\text{cm}}{\text{min}}\right] = 8^2(2)\pi \frac{\text{cm}^2}{\text{min}} = 64(2)\pi \frac{\text{cm}^2}{\text{min}} = 128\pi \ \frac{\text{cm}^2}{\text{min}}.$$

$\mathcal{F}_2$ [Summarize]

Thus, the rate of change of the surface area is 128π. Or $\frac{dSA}{dt} = 128\pi \ \frac{\text{cm}^2}{\text{min}}$.

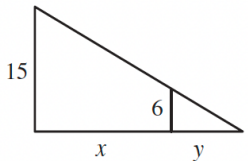
Similar Triangles

QUESTION [1] 3.9.15 (Title of the Essay)

A streetlight is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5ft/s along a straight path. How **fast is the tip of his shadow** moving when he is 40 ft from the pole?

ANSWER [Page A80] (Summary of Essay)

15. (a) The height of the pole (15 ft), the height of the man (6 ft), and the speed of the man (5 ft/s)
(b) The rate at which the tip of the man's shadow is moving when he is 40 ft from the pole
(c)



(d) $\frac{15}{6} = \frac{x+y}{y}$ (e) $\frac{25}{3}$ ft/s

Image from 8th Edition Stewart Calculus [1]

SOLUTION (Intro, Body, and Summary of the Essay)

RELATED RATES Essay Writing

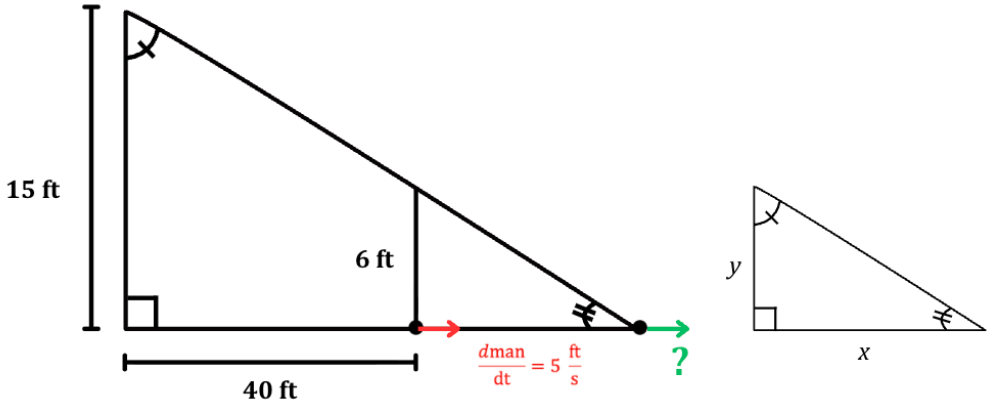
$\mathcal{F}_0$ [collect data] Keywords: fast/tip/shadow(triangle)

Equations Needed	Given Data	Looking for
We are talking about a triangular relation. Pythagorean $x^2 + y^2 = r^2$ Circumference $C = x + y + r$ Area of Right Triangle $A = \frac{1}{2}bh$	$h_{\text{pole}} = 15$ ft $h_{\text{man}} = 6$ ft $\frac{d\text{man}}{\text{dt}} = 5\ \frac{\text{ft}}{\text{s}}$ $d_{\text{pole}\to\text{man}} = 40$ ft	If possible. Else implicit differentiation is necessary.

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[EXAM] For related rates, sometimes the algorithmic approach falls into place without thinking; sometimes you need to think. This problem is likely to be an exam question. It is a question that students struggle with because they need a better trigonometric recollection from their studies, but they do not recall the "similar triangle" relation.

What we have here, is a similar triangle relation—that is, the two triangles (right triangles) in question have the same angles but different side/diagonal-lengths.

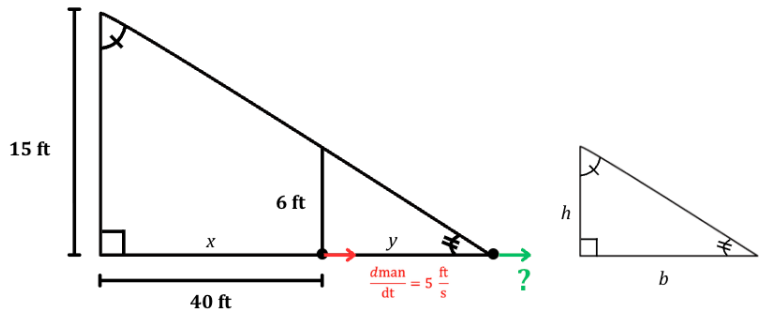


For this question, we are seeking the rate of change of the tip of the shadow. That would be the x coordinate.

For similar triangles, we know that the ratio of the height and base are equal. Thus,

$$\frac{15}{x+40} = \frac{y}{x}$$

Keep in mind that the distance from the pole is changing as well as the distance from the man to the tip of the shadow. So, the previous relation is not going to work because it implies that 40 feet is stationary, which it is not. This is why the author gave extra bits of information in the answer bank. We must account for the distance from the pole to the man as a variable that is changing. We will use the books x and y placement.



For our similar triangle, we have

$$\frac{h_1}{b_1} = \frac{h_2}{b_2} \Leftrightarrow \frac{15}{x+y} = \frac{6}{y}$$

Now, we need to get the equation into a single variable or implicitly differentiate.

$\mathcal{F}_1$ [Implicitly Differentiate w/Respect to Time]

$$\frac{15}{x+y}=\frac{6}{y} \;\Rightarrow\; 15y=6(x+y) \;\Rightarrow\; 15y=6x+6y \;\Rightarrow\; 9y=6x$$

$$\Rightarrow\; y=\frac{2}{3}x \;\Rightarrow\; \frac{d}{dt}\left[y=\frac{2}{3}x\right] \;\Rightarrow\; \frac{dy}{dt}=\frac{2}{3}\frac{dx}{dt}$$

For the smaller triangle, we see that the hypotenuse and the base are changing but the height is not. I would imagine we need to address this situation. Considering the Pythagorean Theorem, $b^2+h^2=r^2$ and implicitly differentiated,

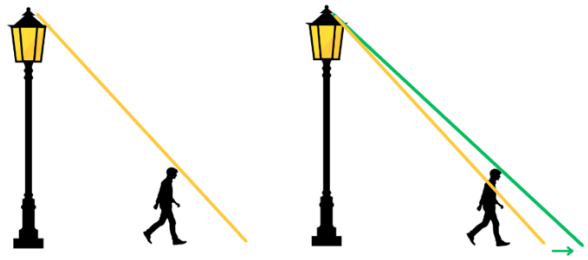
$$2bb'+2hh'+2rr' \;\Rightarrow\; b\frac{db}{dt}+(0)\frac{dh}{dt}=r\frac{dr}{dt} \;\Leftrightarrow\; y\frac{dy}{dt}=r\frac{dr}{dt}$$

From Student X during the live session, they recognized that the rate of change of the shadow is

$$y'(t)=\frac{2}{3}x'(t)=\frac{2}{3}\left(5\frac{\text{ft}}{s}\right)=\frac{10}{3}\frac{\text{ft}}{\text{s}}.$$

Then, the rate of change of the man (given) $5\frac{\text{ft}}{s}$.

Since the shadow is moving at its own rate relative to the height of the man changing the hypotenuse of the smaller triangle, it would make sense that the sum of the rate of the man with the relative rate of the shadow would be the answer. The only question is: what direction is the rate of the shadow moving—left/right?



Take note that the shadow is moving in the positive direction relative to the pole. [RP] If the man was walking towards the pole, would it be subtraction?

$$\left(\frac{d\text{man}}{\text{dt}}=5\ \frac{\text{ft}}{\text{s}}\right)+\left(\frac{dy}{dt}=\frac{10}{3}\ \frac{\text{ft}}{\text{s}}\right)$$

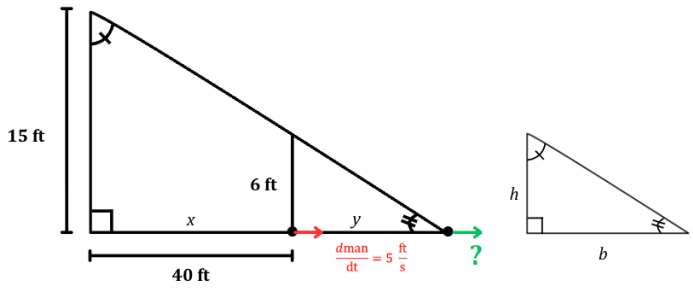
$\mathcal{F}_2$ [Summarize]

Thus, the rate of the tip of the shadow is moving at the rate of the man's speed plus the rate of the shadow expanding (let the shadow be stationary so the tip is moving at rate $y'(t)$).

$$\text{rate of the shadow's tip}=5\ \frac{\text{ft}}{\text{s}}+\frac{10}{3}\ \frac{\text{ft}}{\text{s}}=\frac{(5)(3)+(1)(10)}{(1)(3)}\ \frac{\text{ft}}{\text{s}}=\frac{25}{3}\ \frac{\text{ft}}{\text{s}}.$$

EXAM METHOD (*Practice this until you have memorized and other variations of it by hand.*)

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Imagery is not necessary unless requested by professor.

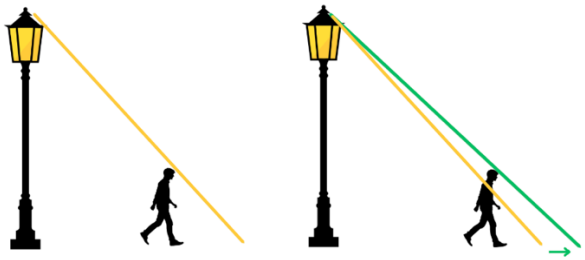
Similar Triangle

$$\frac{h_1}{b_1} = \frac{h_2}{b_2} \Leftrightarrow \frac{15}{x+y} = \frac{6}{y}$$

$$\frac{15}{x+y} = \frac{6}{y} \Rightarrow 15y = 6(x+y) \Rightarrow 15y = 6x + 6y \Rightarrow 9y = 6x$$

$$\Rightarrow y = \frac{2}{3}x \Rightarrow \frac{d}{dt}\left[y = \frac{2}{3}x\right] \Rightarrow \frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt}$$

Imagery is not necessary unless requested.



Take note that the shadow is moving in the positive direction relative to the pole. [RP] If the man was walking towards the pole, would it be subtraction?

$$\left(\frac{d\text{man}}{\text{dt}} = 5\ \frac{\text{ft}}{\text{s}}\right) + \left(\frac{dy}{dt} = \frac{10}{3}\ \frac{\text{ft}}{\text{s}}\right)$$

$$\text{rate of the shadow's tip} = 5\ \frac{\text{ft}}{\text{s}} + \frac{10}{3}\ \frac{\text{ft}}{\text{s}} = \frac{(5)(3)+(1)(10)}{(1)(3)}\ \frac{\text{ft}}{\text{s}} = \frac{25}{3}\ \frac{\text{ft}}{\text{s}}.$$

If you are in college, the first part of this solution is practice for professionality and comprehension. The second part (the exam method) is for you to practice with pencil and paper (provided you have a handwritten proctored exam), many times. Always remember, "Doing homework gets you really good at doing homework. Taking exams will help you get better at exams."—Me.

You should always seek out the other situations in relation to questions done in class or in homework. E.g. (for example), if you have a similar problem with an airplane where the radius is changing. If this question was on the homework, I would anticipate a question with a moving radius/height to be on the exam.

Chapter 1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

QUESTION [1] 1.1.27

Suppose the system below is **consistent** for all possible values of f and g. What can you say about the coefficients of c and d?
Justify your answer.

$$x_1 + 3x_2 = f$$
$$cx_1 + dx_2 = g$$

ANSWER [Pg. A18]

If a linear system is consistent, then the solution is unique if and only if *every column in the coefficient matrix is a pivot column; otherwise, there are infinitely many solutions*.

SOLUTION

STUDENT Linear Algebra is more of an English course than the arithmetic-formula-based math courses you are used to—that is, you cannot do it without knowing all the terminology, definitions, and how they apply to theorems.

"Mathematics is the art of giving the same name to different things."—Henri Poincaré

This is never truer than in linear algebra.

Test taking tip 101: *If you don't know what to do, put it in an augmented matrix and solve.*

$\mathcal{F}_1$ [Formulae] KEYWORDS **consistent**

Before we get started, for linear algebra, much like physics, we are seeking keywords before we begin reading the question. Start by going to the end of the question and working your backward to find keywords.

"A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution."—[Pg. 4]

From the above statement, we simply need to solve the system of equations and identify the values that make the system consistent.

$\mathcal{F}_2$ [Formulate]

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NOTE For book [1], the dot product is not used. Rather, matrix multiplication or 'inner product' notation. You must-must stick with the books notation as it is like using MLA format when you are assigned APA.

$$\begin{matrix} x_1 + 3x_2 = f \\ cx_1 + dx_2 = g \end{matrix} \Rightarrow \begin{matrix} (1)(x_1) + (3)(x_2) = f \\ (c)(x_1) + (d)(x_2) = g \end{matrix} \Rightarrow \begin{matrix} [1 \quad 3]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [f] \\ [c \quad d]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [g] \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & f \\ c & d & g \end{array}\right].$$

At this point, there are many ways to reduce or solve this system. We could go back to prealgebra and simply do substitution, ..., but are we in linear algebra, so we go to the section we are in and look at what is allowed to be used.

ELEMENTARY ROW OPERATIONS [Pg. 6]

1. (Replacement) Replace one row by the sum of itself and multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Now that we know what rules we are allowed to use, we can execute:

NOTE In this book, \sim "tilda" "~" implies an operation has been performed to reduce the matrix.

IMPORTANT When doing row operations, in most cases, you are eliminating in a down column, one column at-a-time, in counterclockwise order to up column when reaching the $m \times n$ element.

$$\begin{bmatrix} 1 \times 1 & \uparrow & \leftarrow \uparrow \\ \downarrow & \ddots & \uparrow \\ \downarrow \rightarrow & \downarrow & m \times n \end{bmatrix}$$

To get an idea of operations, you look at element a_{11} and you ask yourself, "What do I need to multiple entry a_{11} with to add to a_{21} to get zero?" I.e., $[?]a_{11} + a_{21} = 0$.

$$\left[\begin{array}{cc|c} 1 & 3 & f \\ c & d & g \end{array}\right] \quad \begin{matrix} -cR_1 + R_2 \Leftarrow R_2 \\ -c[1 \quad 3 \quad f] + [c \quad d \quad g] \end{matrix} \quad \left[\begin{array}{cc|c} 1 & 3 & f \\ 0 & -3c + d & -cf + g \end{array}\right]$$

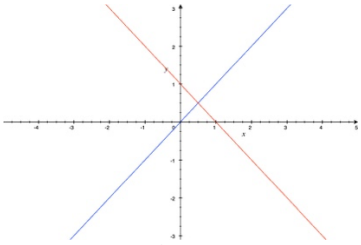
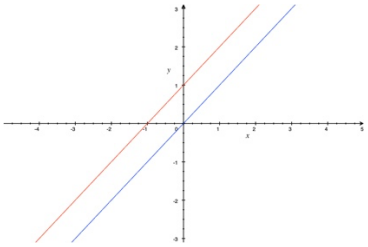
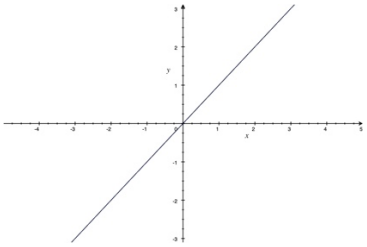
We examine the following, "**consistent** for all possible values of f and g"

$$-3c + d = -cf + g$$

2D Solution

Unique Solution(Consistent)	No Solution(Inconsistent)	Infinite Solution(Consistent)
$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 1 & 0 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 0 & 1 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 0 & 0 \end{array}\right]$

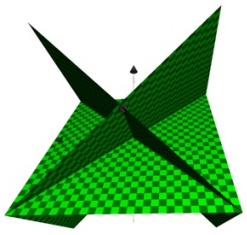
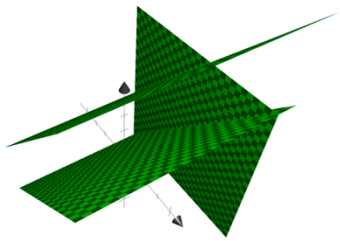
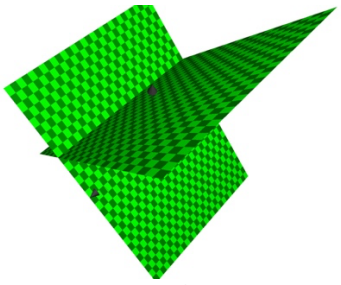
Graphical Application

Unique Solution(Consistent)	No Solution(Inconsistent)	Infinite Solution(Consistent)
 *point of intersection*	 *parallel lines*	 *multiples*

3D Solution

Unique Solution(Consistent)	No Solution(Inconsistent)	Infinite Solution(Consistent)
$\left[\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$

Graphical Application

Unique Solution(Consistent)	No Solution(Inconsistent)	Infinite Solution(Consistent)
 *point of intersection*	 *parallel planes*	 *equation of a line*

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We are looking at the equation for $0 = 0$ and $1 = 0$ in the "consistent" cases per the question's request.

$$-3c + d = -cf + g \;\Rightarrow\; -3c + d = -fc + g$$

We look at the three cases:

[1] For inconsistent, $-3c + d = 0$ and $-fc + g \neq 0$.

[2] For consistent, $-3c + d \neq 0$ and $-fc + g = 0$ or $-3c + d = 0$ and $-fc + g = 0$.

$\mathcal{F}_3$ [Finalize]

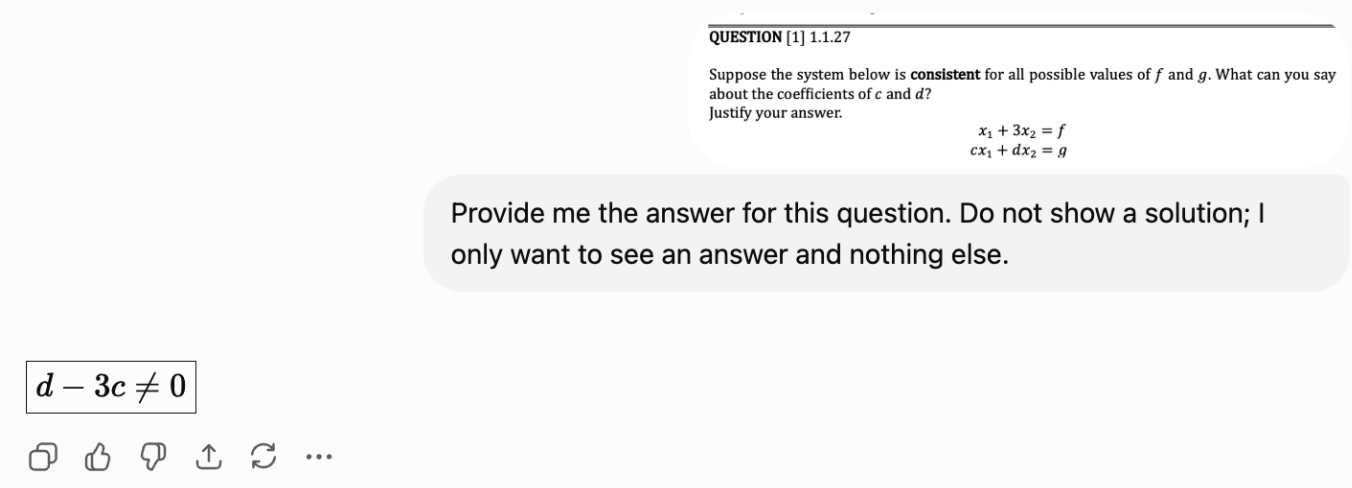
What we can say about c and d is that the system is consistent iff (if and only if, $\Leftrightarrow$ \Leftrightarrow) $d \neq -3c$ when $-fc \neq g$ or $d = -3c$ and $-fc = g$. This gives us the $x_3 = 0$ and $x_3 = free$ cases.

NOTE The answer in the book never addressed c, d. You may want to try and use A.I. technology to compare. That would be a mistake. You should talk with your professor and only your professor. Once you learn how to speak math, you can use A.I. to assist with tedious tasks and discovery by comparison and so forth. To have it help you with math would be wrong at this level. However, I will ask ChatGPT to give me an answer NOT A SOLUTION to show you a comparison.

I will prompt ChatGPT as follows with a screen shot of the question.

Provide me the answer for this question. Do not show a solution; I only want to see an answer and nothing else. (This prompt will prevent plagiarism and cheating.)

ChatGPT Response



In a nutshell: the same answer I got but without explanation(justify). Therefore, ultimately wrong. On an exam, you would receive zero credit for providing the answer ChatGPT provided. DO NOT ASK A.I. for a solution EVER. If you do, leave college; it's not for you!

Introduction to Problem Solving—Analytical

GENERAL(Analytical Problem Solving) ODE—Ordinary Differential Equations is a fairly basic course—that is, for the most part, all you do is apply formulae. Your ability to do intense algebraic operation in relation to integration and differentiation is key. The rest is formulae application. However, for engineers, physicists, and applied mathematicians, you may find yourself working on physics problems.

PHYSICISICTS(Applied ODE) Note to all STEM(PLEM) majors, at this level of ODE(first semester ODE), the application of the math to physics does not require you to understand the physics. You simply need to know the formula and where to plug the constants in, then you apply the analytical formulaic approach to problem solving. You will find this in equilibrium problems related to fluid and spring problems...

ENGINEERS(Applied ODE) For engineers (generally speaking), as an ME or EE (Mechanical Engineer or Electrical), you will most likely take the same ODE course as the physics majors and math majors. As an aerospace engineer or civil or something in that world, you most likely take an accelerated combination of Differential Equations fused with Linear Algebra. This requires mostly a lot of reading and problem solving. Work closely with the professor in lecture and write down everything they write down as that is what is likely to be the exam.

MATHEMATICIANS(Applied ODE) You will take the standard ODE course, likely with the engineers and physicists. In junior/senior year, you might dive deeper into the analysis side. Mostly though, ODE and PDE(Partial Differential Equations), is for applied mathematicians as differential equations are what model pretty everything, we deal with in 3D space.

Differential Equations with Linear Algebra: For those of you who do not have a formal linear algebra course separate from differential equations, this course will be very difficult. However, it is not hard—that is, if you strategize rather than try to comprehend. There are two sides to college: The side where you learn and the side you strategize. You learn to think like the professor.

How to approach problem solving in ODE

QUESTION

[i] Title of Essay

ANSWER

Summary of Essay with Source

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SOLUTION

Body of the Essay

$\mathcal{F}_0$ [Formulas, Definitions, Equations, Rules, Properties, ...] 50% Of Exam Credit.

CLASSIFICATION

[ii] Introduction of Essay with source

$\mathcal{F}_1$ [Formulation of $\mathcal{F}_0$] 50% of Exam Credit – 0% Without $\mathcal{F}_0$.

[iii] Body of Essay

$\mathcal{F}_2$ [Finalize] Clearly state the answer with reference. 0% of Exam Credit.

[iv] Summary of Essay with Source

EXAM METHOD

This portion is to show how to shorthand for exam solutions to get full credit and maximized time—allotted.

[RP] [FC (Full Circle | Research Position)] Show the answer satisfied the question—that is, prove your answer is the correct answer

To the college student:

When you have a question, and you get the answer. Big deal...a smartphone can do that. When you can provide a detailed algorithm of how to get the answer every time as well as a secondary algorithm that shows the answer is in fact correct without a doubt, then you are doing math correctly.

If your attitude for what I just wrote is, "No body told me that.", you are in the wrong place. Scientists solve problems, they don't blame other people for their inability to take ownership of their own education.

The Algorithmic Approach to Solving ODE's

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*EXAM*QUESTION [1] 8.5.7 Cauchy–Euler (Equidimensional) Equations

Use the substitution $y = x^r$ to find a general solution to the given equation for $x > 0$.

$$x^3y'''(x) + 4x^2y''(x) + 10xy'(x) - 10y(x) = 0$$

ANSWER [Pg. B-19]

$$c_1 x + c_2 x^{-1} \cos(3 \ln x) + c_3 x^{-1} \sin(3 \ln x)$$

SOLUTION

$\mathcal{F}_0$ [Formulas, Definitions, Equations, Rules, Properties, ...] 50% Of Exam Credit.

This question has third order derivative. The book does not necessarily address this.

When solving an ODE, the first thing you do is classify it.

CLASSIFICATION Third-Order-Linear-Homogenous-ODE (Cauchy-Euler) Why?

1) Third order because the highest derivative is of order 3:

$$x^3 y''' + 4x^2 y'' + 10xy' - 10y = 0$$

2) Linear because there are no operators operating on the dependent variable:

$$y' = \frac{dy \leftarrow \text{dependent variable}}{dx \leftarrow \textit{in}\text{dependent variable}}$$

$$\text{NL: } y' - \frac{x}{y} = e^y, \qquad \text{L: } y' + \frac{y}{x} = y$$

3) Homogeneous because there are no functions consisting solely of the independent variable. E.g.,

$$\text{NH: } y' - x = e^y, \qquad \text{H: } y' + \frac{y}{x} = \sin y$$

CAUCHY-EULER METHOD

The Aglorithm: Let $y = x^r$.

Repeated Zeroes $r = \{r_1, r_1, \dots\}$

$$y = x^r, \qquad y = x^r + x^r \ln x, \qquad \left[\text{[RP] } y = x^r + x^r \ln x + \frac{1}{2} x^r \ln^2 x, \dots ?\right]$$

Two Real Zeroes $r = \{r_1, r_2\}$

$$y_1 = x^{r_1}, \qquad y_2 = x^{r_2}$$

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Complex Zeroes $r = \{\alpha \pm \beta i\}$

$$y_1 = x^{\alpha}\cos(\beta \ln x), \qquad y_2 = x^{\alpha}\sin(\beta \ln x)$$

$\mathcal{F}_1$ [Formulation of $\mathcal{F}_0$] 50% of Exam Credit – 0% Without $\mathcal{F}_0$.

Now, we can assemble everything from the algorithm. Start with $y = x^r$.

$$y = x^r, \qquad y' = rx^{r-1}, \qquad y'' = r(r-1)x^{r-2}, \qquad y''' = r(r-1)(r-2)x^{r-3}.$$

$$x^3y''' + 4x^2y'' + 10xy' - 10y = 0$$

$$\Rightarrow \quad x^3[r(r-1)(r-2)x^{r-3}] + 4x^2[r(r-1)x^{r-2}] + 10x[rx^{r-1}] - 10[x^r] = 0$$

$$\Rightarrow \quad [r(r-1)(r-2)x^{r-3+3}] + 4[r(r-1)x^{r-2+2}] + 10[rx^{r-1+1}] - 10[x^r] = 0$$

$$\Rightarrow \quad [r(r-1)(r-2)x^r] + 4[r(r-1)x^r] + 10[rx^r] - 10[x^r] = 0$$

$$\Rightarrow \quad x^r(r(r-1)(r-2) + 4r(r-1) + 10r - 10) = 0$$

$$\Rightarrow \quad x^r > 0, \qquad r(r-1)(r-2) + 4r(r-1) + 10r - 10 = 0$$

$$\Rightarrow \quad r(r^2 - 3r + 2) + 4r^2 - 4r + 10r - 10 = 0$$

$$\Rightarrow \quad r^3 - 3r^2 + 2r + 4r^2 - 4r + 10r - 10 = 0$$

$$\Rightarrow \quad r^3 + r^2 + 8r - 10 = 0$$

How do we solve this? By factoring? Let's look at the **[RP] BLANK Method** ... (for researchers, go find the name and theorem for the following method. I cannot remember the name off of the top of my head; I remember the technique.)

We look at the coefficient of the leading term 1 and the coefficient of the scalar/constant trailing term a_0x^0 of the polynomial, 10. We then take the ratio of the multiples of these $\{\pm 1\}; \{\pm 1 \ \pm 2, \pm 5, \pm 10\}$. We then test by inspection or rigorous computation which of the set of ratios.

Start with the number 1: $r^3 + r^2 + 8r - 10 = (\pm 1)^3 + (\pm 1)^2 + 8(\pm 1) - 10 = 0$.

We chose wisely, +1 is the correct zero—that is, $r = 1 \Rightarrow (r - 1) = 0$

$$(1)^3 + (1)^2 + 8(1) - 10 = 1 + 1 + 8 - 10 = 10 - 10 = 0.$$

Now, we divide $(r^3+r^2+8r-10)/(r-1)$: You can divide with whatever method you like at this point of calculus(ODE)—i.e., long division, synthetic division, decomposition, ...

$$\frac{r^3+r^2+8r-10}{r-1} \quad \Rightarrow \quad r-1 \;\overline{|\, r^3+r^2+8r-10} \quad r^2+2r+10$$

$$\begin{array}{r} -(r^3-r^2+(0)r+0) \\ \hline +0+2r^2+8r-10 \\ \hline -(2r^2-2r+0) \\ \hline +0+10r-10 \\ \hline -(10r-10) \\ \hline 0 \end{array}$$

$$\therefore \;\; (r-1)(r^2+2r+10)=0, \qquad b^2-4ac=4-4(10)<0 \therefore r\in\mathbb{C}.$$

$$r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{(2)^2-4(1)(10)}}{2(1)}=\frac{-2\pm\sqrt{-36}}{2}$$

$$=-\frac{2}{2}\pm\frac{\sqrt{-6^2}}{2}=-1\pm\frac{6}{3}\sqrt{-1}=-1\pm 3i=\alpha\pm\beta i. \quad \therefore \alpha=-1, \qquad \beta=3.$$

$$\therefore y_1=x, \qquad y_2=x^{-1}\cos(3\ln x), \qquad y_3=x^{-1}\sin(3\ln x).$$

$\mathcal{F}_2$ [Finalize] Clearly state the answer with reference. 0% of Exam Credit.

Therefore, by "Cauchy-Euler Method," we see that

$$y=c_1y_1+c_2y_2+c_3y_3=c_1x+c_2x^{-1}\cos(3\ln x)+c_3x^{-1}\sin(3\ln x).$$

EXAM METHOD **This is definitely 99% likely to be on an exam!**

By Cauchy-Euler Method: Let $y=x^r$.

$$y=x^r, \qquad y'=rx^{r-1}, \qquad y''=r(r-1)x^{r-2}, \qquad y'''=r(r-1)(r-2)x^{r-3}.$$

$$x^3y'''+4x^2y''+10xy'-10y=0$$

$$\Rightarrow \quad x^3[r(r-1)(r-2)x^{r-3}]+4x^2[r(r-1)x^{r-2}]+10x[rx^{r-1}]-10[x^r]=0$$

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$$\Rightarrow \quad [r(r-1)(r-2)x^{r-3+3}] + 4[r(r-1)x^{r-2+2}] + 10[rx^{r-1+1}] - 10[x^r] = 0$$

$$\Rightarrow \quad [r(r-1)(r-2)x^r] + 4[r(r-1)x^r] + 10[rx^r] - 10[x^r] = 0$$

$$\Rightarrow \quad x^r(r(r-1)(r-2) + 4r(r-1) + 10r - 10) = 0$$

$$\Rightarrow \quad x^r > 0, \qquad r(r-1)(r-2) + 4r(r-1) + 10r - 10 = 0$$

$$\Rightarrow \quad r(r^2 - 3r + 2) + 4r^2 - 4r + 10r - 10 = 0$$

$$\Rightarrow \quad r^3 - 3r^2 + 2r + 4r^2 - 4r + 10r - 10 = 0 \quad \Rightarrow \quad r^3 + r^2 + 8r - 10 = 0$$

$$\{\pm 1\}; \{\pm 1 \ \pm 2, \pm 5, \pm 10\}$$

$$r^3 + r^2 + 8r - 10 = (\pm 1)^3 + (\pm 1)^2 + 8(\pm 1) - 10 = 0 \ldots$$

$$\text{Clearly,} \qquad r = 1 \ \Rightarrow \ (r-1) = 0.$$

$$\frac{r^3 + r^2 + 8r - 10}{r - 1} \quad \Rightarrow \quad r - 1 \quad \overline{|\, r^3 + r^2 + 8r - 10} \quad r^2 + 2r + 10$$

$$\begin{array}{r} -(r^3 - r^2 + (0)r + 0) \\ \hline +\ 0 + 2r^2 + 8r - 10 \\ \hline -(2r^2 - 2r + 0) \\ \hline +\ 0 + 10r - 10 \\ \hline -(10r - 10) \\ \hline 0 \end{array}$$

$$\therefore \quad y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 x + c_2 x^{-1} \cos(3 \ln x) + c_3 x^{-1} \sin(3 \ln x).$$

[RP] [FC (Full Circle | Research Position)] Show the answer satisfied the question—that is, prove your answer is the correct answer. HINT Take the derivative!

To the college student:

When you have a question, and you get the answer. Big deal...a smartphone can do that. When you can provide a detailed algorithm of how to get the answer every time as well as a secondary algorithm that shows the answer is in fact correct without a doubt, then you are doing math correctly.

If your attitude for what I just wrote is, "No body told me that.", you are in the wrong place. Scientists solve problems, they don't blame other people for their inability to take ownership of their own education.

Quantum Mechanics

What is quantum mechanics? I think this is the best place for any scientist to start. What is quantum mechanics? I think, first, we should ask ourselves: what is the definition of *quantum* and what is the definition of *mechanics*?

Quantum Mechanics: "A subject no one knows anything about..."—@authorjond

Quantum "a discrete quantity of energy proportional in magnitude to the frequency of the radiation it represents"—Oxford Languages

Mechanics "the branch of **applied mathematics** dealing with motion and forces producing motion"—Oxford Languages

Let us begin with understanding English (or your native language).

What is English? "the language of England, widely used in many varieties throughout the world" —Oxford Languages

Quantum:

- a
  - used to indicate membership of a class of people or things.
- discrete
  - individually separate and distinct.
- quantity
  - the amount or number of a material or immaterial thing not usually estimated by spatial measurement
    - spatial: relating to or occupying space.
- of
  - expressing the relationship between a part and a whole
- energy
  - power derived from the utilization of physical or chemical resources, especially to provide light and heat or to work machines
- proportional
  - (of a variable quantity) having a constant ratio to another quantity
- in
  - expressing a period of time during which an event takes place or a situation remains the case

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- magnitude
    - size
- to
    - approaching or reaching
- the
    - denoting one or more people or things already mentioned or assumed to be common knowledge
- frequency
    - the rate at which a vibration occurs that constitutes a wave, either in a material (as in sound waves), or in an electromagnetic field (as in radio waves and light), usually measured per second
- of
    - expressing the relationship between a part and a whole
- the
    - denoting one or more people or things already mentioned or assumed to be common knowledge
- radiation
    - radiation
- it
    - used to refer to a thing previously mentioned or easily identified
- represents
    - constitute, amount to

Make sure you understand that definitions of words are not written by God. They are subjective and precarious.

Precarious "not securely held or in position; dangerously likely to fall or collapse"

The purpose of this exercise is to explain to you that you don't actually know what you are talking about when you are talking. Rather, you communicate to survive. It is a chain reaction that is uncontrollable—that is, communication.

Now, let us begin our journey into the mathematical—i.e., the symbolic representation of nonsensical nonsense! What I mean is: we may not understand physics at all, but we can sure as hell explain it mathematically even though we do not understand math at all. How could we? We don't even know the definitions of the words we are speaking.

Now, from the former literature do you know what quantum mechanics is? Let's ask the internet

quantum mechanics the branch of mechanics that deals with the mathematical description of the motion and interaction of subatomic particles, incorporating the concepts of quantization of energy, wave-particle duality, the uncertainty principle, and the correspondence principle

I think the biggest issue students have with physics (in general) is simply understanding what a particle is. What is a particle?

In physics, a particle is an object with no mass. Oxford languages- "1. a minute portion of matter. 2. the least possible amount."

IMPORTANT I encourage you read the former before we math. No one understands physics; we are able to explain phenomena utilizing language(math) is all. We are able to understand explaining physics with the language of math with our trained language(s) and our body language and the use of our body via physical forces. This is important as you will never be able to understand anything, you will only learn how to describe phenomena via mathematics.

The review of many books on Quantum Mechanics points to the Schrödinger Equation and the wave equation as a starting point. For me personally, after 20-years of study, still wonder what the hell Quantum Mechanics is. I think of it as the art of guessing but experimental physicists would think very different. Maybe they see things differently—physically. I can only (I feel) see things mathematically.

"Probability is not math; it is the dirty math."—@authorjond

In order to do Quantum Mechanics, we must get dirty!...

I think that the best place to start with QM is basic probabilistic physics. Why? The generalized concept of QM is based on chance. Fascinating really as one's own prayers can affect the outcome of calculation. This means that QM is fictitious in nature and one's own fantasy of reality can dictate the outcome via probability. Do you understand? QM is essentially prayer as it is statistically probabilistically sometimes going to work and the outcome is highly based on the calculator's desired outcome and if one positions themselves in a position of portraying outcome, others tend to bend to the outcome.

Schrödinger Equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi$$

$$i\hbar u_t = -\frac{\hbar^2}{2m}u_{xx} + Vu$$

Wave Function

$$\psi(x,t)$$

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Power Series Solutions and Special Functions

30 Regular Singular Points (Continued)

QUESTION [32] 30.2

The equation

$$4x^2y'' - 8x^2y' + (4x^2 + 1)y = 0$$

Has *only one* **Frobenius** series solution. Find the **general solution**.

ANSWER

$$y = c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x$$

SOLUTION

[**READER***] The next ten pages is an attempt to solve the problem with standard series solution techniques. It is in essence gibberish and leads nowhere, however, the math is sound more or less for other situations, so it is worth reviewing. Jump to the formula, derived from the gibberish ten pages from now!

Begin with keywords where necessary: *only one* **Frobenius**, **general solution**

A general solution contains constants where a particular solution solve, with an initial value to find the constants.

NOTE In this text, the author chooses to define $\ln x \equiv \log x$.

Frobenius Form

$$y = x^m(a_0 + a_1x + a_2x^2 + \cdots) = x^m \sum_{n=0}^{\infty} a_n x^n$$

"it is much simpler to substitute the general Frobenius series $y = x^m(a_0 + a_1x + a_2x^2 + \cdots)$ directly into the differential equation than to use the recursion formula"—Pg. 232

Theorem A

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n, \qquad y_2 = x^{m_2} \sum_{n=0}^{\infty} a_n x^n, \qquad a_0 \neq 0$$

From form, $y'' + P(x)y' + Q(x)y = 0$. Then, from 'hypothesis,' $xP(x)$ and $x^2Q(x)$ are analytical at $x = 0$.

After reviewing the section during the live session, we see that

$$y_1 = x^{m_1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{m_1+n}$$

From the book's suggestion to use the non-summation form as the easiest approach, we will begin there. First get the ODE in the correct form.

$$4x^2y'' - 8x^2y' + (4x^2+1)y = 0 \;\Rightarrow \frac{1}{4x^2}[4x^2y'' - 8x^2y' + (4x^2+1)y = 0]$$

$$\Rightarrow\; y'' - 2y' + \frac{4x^2+1}{4x^2}y = 0 \;\Rightarrow\; P(x) = -2, \qquad Q(x) = \frac{4x^2+1}{4x^2}.$$

Now,

$$y = x^m(a_0 + a_1x + a_2x^2 + \cdots) \;\Rightarrow\; y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots$$

$$\Rightarrow\; y' = a_0mx^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots$$

$$\Rightarrow\; y'' = a_0m(m-2)x^{m-2} + a_1m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m + \cdots$$

Or,

$$y = x^m\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$y' = \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1}, \qquad y'' = \sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2}$$

$$y_2 = y_1\log x + x^{m_2}\sum_{n=0}^{\infty} c_n x^n$$

From reading the section, I believe that there is a simple solution that is formula based. At this point, I am deriving the algorithm in laymen terms.

What I am seeing is an end result of solving for m_1 and m_2. Since the question clearly states that there a single solution. Thus, from CASE A Pg. 233, $m_1 = m_2$ yields the recursion formula

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$$a_n f(m_2+n) = -a_0(m_2 p_n + q_n) - \cdots - a_{n-1}[(m_2+n-1)p_1 + q_1]$$

NOTE CASE A, B, and C all address no second solution.

The book suggests, $y_1 = x^m(a_0 + a_1 x + a_2 x^2 + \cdots)$ and with the method found in section 16, we see that $y_2 = v y_1$.

$$y_2 = y_1 \log x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n$$

In order to find y_1, we need to solve for m and $\{a_0, a_1, a_2, \ldots\}$. Then, we are home free.

According to the text, $m(m-1) + m p_0 + q_0 = 0$. We need to find p_0 and q_0

To find p_0, q_0, we look to the following equations

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

Then,

$$P(x) = -2, \qquad Q(x) = \frac{4x^2+1}{4x^2} \;\Rightarrow\; x(-2) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2\left(\frac{4x^2+1}{4x^2}\right) = \sum_{n=0}^{\infty} q_n x^n$$

$$\Rightarrow\; -2 = \sum_{n=0}^{\infty} p_n x^{n-1}, \qquad \left(\frac{4x^2+1}{4}\right) = \sum_{n=0}^{\infty} q_n x^n$$

$$\Rightarrow\; -2 = \sum_{n=0}^{\infty} p_n x^{n-1}, \qquad 4x^2+1 = 4\sum_{n=0}^{\infty} q_n x^n$$

$$\Rightarrow\; -2 = \sum_{n=0}^{\infty} p_n x^{n-1}, \qquad 4x^2 = 4\sum_{n=0}^{\infty} q_n x^n - 1$$

$$\Rightarrow\; 1 = -\frac{1}{2}\sum_{n=0}^{\infty} p_n x^{n-1}, \qquad 1 = \sum_{n=0}^{\infty} q_n x^{n-2} - \frac{1}{4x^2}$$

We are looking for p_0 and q_0. Can we do this from the above?

$$1 = -\frac{1}{2}\sum_{n=0}^{\infty} p_n x^{n-1} =$$

$$1 = -\frac{1}{2}\sum_{n=0}^{\infty} p_n x^{n-1} = \frac{1}{2}\left[\frac{p_0}{x} + p_1 + p_2 x + p_3 x^2 + \cdots\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{p_0}{x} + p_1 + p_2 x + p_3 x^2 + \cdots\right] = 1$$

$$\Rightarrow \left[\frac{p_0}{x} + p_1 + p_2 x + p_3 x^2 + \cdots\right] =$$

From the text, $m_2 = 1 - p_0 - m_1$. Since $m_1 = m_2$, $m_1 = 1 - p_0 - m_1 \Rightarrow m_1 = \frac{1}{2}(1 - p_0)$

p_0 is clearly equal to 0 but how do we show this and the summation outputs $e^x = \sum x^k/k!$?

The text also defines $k = m_1 - m_2 + 1$, which suggests $k = 1$.

Recursion

I think, at this point, the author wishes for us to find $\{a_0, a_1, a_2, \ldots\}$ and then, we can plug in to the formulae.

Let us plug and chug!

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$$y = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$y' = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1} = m a_0 x^{m-1} + \sum_{n=1}^{\infty} (m+n) a_n x^{m+n-1} = m a_0 x^{m-1} + \sum_{n=0}^{\infty} \big(m + (n+1)\big) a_{n+1} x^{m+n}$$

$$y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$$

$n = 0$	$m(m-1)a_0 x^{m-2} + \sum_{n=1}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$
$n = 1$	$m(m-1)a_0 x^{m-2} + (m+1)m a_1 x^{m-1} + \sum_{n=2}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$
$n = 0$	$m(m-1)a_0 x^{m-2} + (m+1)m a_1 x^{m-1} + \sum_{n=0}^{\infty} \big(m + (n+2)\big)(m+n+1) a_2 x^{m+n}$

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Now that the series have matching powers, we can plug into the equation and find recursive relation. (I am not convinced this is what we are supposed to do as the text implies simpler path but I am not seeing a formula leading to simplicity—yet.)

$$4x^2y'' - 8x^2y' + (4x^2 + 1)y = 0 \Rightarrow$$

$$4x^2m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + \sum_{n=0}^{\infty}\big(m + (n+2)\big)(m+n+1)a_2x^{m+n}$$

$$-8x^2ma_0x^{m-1} + \sum_{n=0}^{\infty}\big(m + (n+1)\big)a_{n+1}x^{m+n}$$

$$+(4x^2 + 1)\sum_{n=0}^{\infty}a_nx^{m+n} = 0$$

$$(4x^2m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1}) + (-8x^2ma_0x^{m-1}) +$$

$$\sum_{n=0}^{\infty}\big(m + (n+2)\big)(m+n+1)a_nx^{m+n} + \sum_{n=0}^{\infty}\big(m + (n+1)\big)a_{n+1}x^{m+n} + \sum_{n=0}^{\infty}a_nx^{m+n} = 0 \Rightarrow$$

$$(4x^2m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1}) + (-8x^2ma_0x^{m-1}) +$$

$$\sum_{n=0}^{\infty}\big(m + (n+2)\big)(m+n+1)a_nx^{m+n} + \sum_{n=0}^{\infty}\big(m + (n+1)\big)a_{n+1}x^{m+n} + (4x^2 + 1)\sum_{n=0}^{\infty}a_nx^{m+n} = 0 \Rightarrow$$

$$(4x^2m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1}) + (-8x^2ma_0x^{m-1}) +$$

$$\sum_{n=0}^{\infty}\big(m + (n+2)\big)(m+n+1)a_nx^{m+n} + \sum_{n=0}^{\infty}\big(m + (n+1)\big)a_{n+1}x^{m+n} + 4\sum_{n=0}^{\infty}a_nx^{m+n+2} + \sum_{n=0}^{\infty}a_nx^{m+n} = 0 \Rightarrow$$

$$(4x^2m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1}) + (-8x^2ma_0x^{m-1}) +$$

$$\sum_{n=0}^{\infty}\big(m + (n+2)\big)(m+n+1)a_nx^{m+n} + \sum_{n=0}^{\infty}\big(m + (n+1)\big)a_{n+1}x^{m+n} + 4\sum_{n=0}^{\infty}a_nx^{m+n+2} + \sum_{n=0}^{\infty}a_nx^{m+n} = 0 \Rightarrow$$

$$4\sum_{n=0}^{\infty}a_nx^{m+n+2} = 4\sum_{n=2}^{\infty}a_{n-2}x^{m+n+2-2} = 4\sum_{n=2}^{\infty}a_{n-2}x^{m+n}$$

$$4\sum_{n=0}^{\infty} a_n x^{m+n+2} = 4a_0 x^{m+2} + 4\sum_{n=1}^{\infty} a_n x^{m+n+2} = 4a_0 x^{m+2} + a_1 4x^{m+3} + 4\sum_{n=2}^{\infty} a_n x^{m+n+2}$$

$$= 4a_0 x^{m+2} + a_1 4x^{m+3} + 4\sum_{n=4}^{\infty} a_{n-2} x^{m+n}$$

At this point, to get the relation in order, we must expand the other series to $n = 2$. Thus,

$$(4x^2 m(m-1)a_0 x^{m-2} + (m+1)ma_1 x^{m-1}) + (-8x^2 ma_0 x^{m-1}) +$$

$$\sum_{n=0}^{\infty}\big(m+(n+2)\big)(m+n+1)a_n x^{m+n} + \sum_{n=0}^{\infty}\big(m+(n+1)\big)a_{n+1}x^{m+n} + 4\sum_{n=2}^{\infty} a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0 \Rightarrow$$

• Term 1 to $n = 2$

$$(m+2)(m+1)a_0 x^m + (m+3)(m+2)a_1 x^{m+1} + \sum_{n=2}^{\infty}\big(m+(n+2)\big)(m+n+1)a_2 x^{m+n}$$

• Term 2 to $n = 2$

$$(m+1)a_1 x^m + (m+2)a_2 x^{m+1} + \sum_{n=2}^{\infty}\big(m+(n+1)\big)a_{n+1}x^{m+n}$$

• Term 4 to $n = 2$

$$a_0 x^m + a_1 x^{m+1} + \sum_{n=2}^{\infty} a_n x^{m+n}$$

$$(4x^2 m(m-1)a_0 x^{m-2} + (m+1)ma_1 x^{m-1}) + (-8x^2 ma_0 x^{m-1}) + (m+2)(m+1)a_0 x^m$$
$$+ (m+3)(m+2)a_1 x^{m+1} + (m+1)a_1 x^m + (m+2)a_2 x^{m+1} + a_0 x^m + a_1 x^{m+1} +$$

$$\sum_{n=2}^{\infty}\big(m+(n+2)\big)(m+n+1)a_n x^{m+n} + \sum_{n=2}^{\infty}\big(m+(n+1)\big)a_{n+1}x^{m+n} + 4\sum_{n=2}^{\infty} a_n x^{m+n} + \sum_{n=2}^{\infty} a_n x^{m+n} = 0 \Rightarrow$$

$$(4x^2 m(m-1)a_0 x^{m-2} + (m+1)ma_1 x^{m-1}) + (-8x^2 ma_0 x^{m-1}) + (m+2)(m+1)a_0 x^m$$
$$+ (m+3)(m+2)a_1 x^{m+1} + (m+1)a_1 x^m + (m+2)a_2 x^{m+1} + a_0 x^m + a_1 x^{m+1} +$$

$$\sum_{n=2}^{\infty}\Big[\big(m+(n+2)\big)(m+n+1)a_n + \big(m+(n+1)\big)a_{n+1} + 4a_n + a_n\Big]x^{m+n} = 0 \Rightarrow$$

Now, we have the equation to look for the relation. If the above is zero, then $0 + \sum 0 = 0$

$$\big(m+(n+2)\big)(m+n+1)a_n + \big(m+(n+1)\big)a_{n+1} + 4a_n + a_n = 0$$

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And

$$(4x^2m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1}) + (-8x^2ma_0x^{m-1}) + (m+2)(m+1)a_0x^m \\ + (m+3)(m+2)a_1x^{m+1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + a_0x^m + a_1x^{m+1} = 0$$

Start with

$$\big(m+(n+2)\big)(m+n+1)a_n + \big(m+(n+1)\big)a_{n+1} + 4a_n + a_n = 0$$

$$\Rightarrow\ a_n\Big(\big(m+(n+2)\big)(m+n+1)+4\Big) = -\big(m+(n+1)\big)a_{n+1}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = -\frac{\big(m+(n+2)\big)(m+n+1)+4}{m+(n+1)}$$

Let $n = 0$. Then,

$$a_0\big((m+2)(m+1)+4\big) = -(m+1)a_1$$

At this point, we now have to go to previous sections to verify the methods used to find $\{a_0, a_1, a_2, \dots\}$ or try to see if there is a clearer relation in the chapter.

REATTEMPT

Based on a formula, we simply need to find m and $\{a_0, a_1, a_2, \dots\}$. The reading is suggesting a formula. But it is not devised.

From the textbook, the indicial equation is $m(m-1) + mp_0 + q_0 = 0$. If we know what p_0 and q_0 are, we can find m. But what about $\{a_0, a_1, a_2, \dots\}$?

Also, $m_2 = 1 - p_0 - m_1$ and we know $m_1 = m_2$. Thus, $m_1 = \frac{1}{2}(1-p_0)$

Because the answer is provided, it is clear that $p_0 = 0$. We need to show this.

Going back to $P(x)$ and $Q(x)$,

$$1 = -\frac{1}{2}\sum_{n=0}^{\infty} p_n x^{n-1} = -\frac{1}{2}[p_0x^{-1} + p_1 + p_2x + p_3x^2 + \cdots]$$

$$1 = \sum_{n=0}^{\infty} q_n x^{n-2} - \frac{1}{4x^2} = [q_0x^{-2} + q_1x^{-1} + q_2 + q_3x + q_4x^2 + \cdots] - \frac{1}{4x^2}$$

If we can find p_0, we can solve for m. Then, we need to find $\{a_0, a_{-1}, a_2, \dots\}$

REATTEMPT

I am beginning to see the solution. The textbook clearly states the recursion as

$$a_n[(m+n)(m+n-1)+(m+2)p_0+q_0]+\sum_{k=0}^{n-1}a_k[(m+k)p_{n-k}+q_{n-k}]=0.$$

We then know, $a_n[(m+n)(m+n-1)+(m+2)p_0+q_0]=0$
From the text, the above simplifies to $m(m-1)+mp_0+q_0=0$. We also know, $m_1=m_2$ because the question told us there is one Frobenius solution.

Also, $m_2=1-p_0-m_1$ and we know $m_1=m_2$. Thus, $m_1=\frac{1}{2}(1-p_0)$

Thus, if we can find p_0 and q_0, we can then find m_1 and m_2. And then, we can plug in the values for

$$a_n[(m+n)(m+n-1)+(m+2)p_0+q_0]=0$$

to find $\{a_n\}$.

Potential Formula

1) Find p_0 and q_0.
2) Then, find m.
3) Then, find $\{a_n\}$.

From the text

$$P(x)y'=x^{m-2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}p_{n-k}a_k(m+k)+p_0a_n(m+n)\right]x^n$$

$$Q(x)y=x^{m-2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n-1}q_{n-k}a_k+q_0a_n\right]x^n$$

How do we extract the p_0 and q_0?

$$4x^2y''-8x^2y'+(4x^2+1)y=0 \quad\Leftrightarrow\quad y''+P(x)y'+Q(x)y=0$$

$$\Leftrightarrow\quad y''-2y'+\frac{4x^2+1}{4x^2}y=0 \quad\Rightarrow\quad P(x)=-2, \qquad Q(x)=1+4x^{-2}.$$

Then,

$$(-2)y'=x^{m-2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}p_{n-k}a_k(m+k)+p_0a_n(m+n)\right]x^n,$$

$$(1+4x^{-2})y=x^{m-2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n-1}q_{n-k}a_k+q_0a_n\right]x^n.$$

And,

$$y=\sum_{n=0}^{\infty}a_nx^{m+n}, \qquad y'=\sum_{n=0}^{\infty}a_n(m+n)x^{m+n-1}.$$

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Then,

$$(-2)\sum_{n=0}^{\infty} a_n(m+n)x^{m+n-1} = x^{m-2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} p_{n-k}a_k(m+k) + p_0 a_n(m+n)\right]x^n,$$

$$(1+4x^{-2})\sum_{n=0}^{\infty} a_n x^{m+n} = x^{m-2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n-1} q_{n-k}a_k + q_0 a_n\right]x^n.$$

From here, if I can get the left side of the equation to match the right side, we can find p_0 and q_0.

If p_0 is zero, then $m_1 = \frac{1}{2}(1-p_0) = \frac{1}{2} = m_2$. ← is this correct? Because it conflicts with other cases.

The recursive relation

$$a_n f(m_2+n) = -a_0(m_2 p_n + q_n) - \cdots - a_{n-1}[(m_2+n-1)p_1 + q_1].$$

Then,

$$a_n f(m+n) = -a_0(mp_n + q_n) - \cdots - a_{n-1}[(m+n-1)p_1 + q_1].$$

[BREAK] From the live session, I am currently stumped on deriving the formula. Wel shall take a break and think and return. I know it is tempting to use A.I. or other books/solution manuals but that will defeat the purpose of being in college and ruin your education as the struggle we are experiencing right now, is the struggle we are to overcome, in order to become effective scientists!

Right now, we are stumped with identifying p_0 and q_0 from the book.

Since we are stuck, I go back to the previous section to see

$$y'' + \frac{p}{x}y' + \frac{q}{x^2}y = 0$$

$$4x^2y'' - 8x^2y' + (4x^2+1)y = 0 \quad \Leftrightarrow \quad y'' + P(x)y' + Q(x)y = 0$$

$$\Leftrightarrow \quad y'' - 2y' + \frac{4x^2+1}{4x^2}y = 0 \quad \Rightarrow \quad P(x) = -2, \qquad Q(x) = 1 + 4x^{-2}.$$

. If that is the case, $m(m-1) + mp_0 + q_0 = 0$

REATTEMPT TYPO ↗

It now appears that p_0 and q_0 can be easily identified by a formula similar to the previous Cauchy-Euler looking scenarios $x^2y'' + xy + y = 0$.

$$y'' - 2y' + \frac{4x^2+1}{4x^2}y = 0$$

$$P(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + \cdots = -2.$$

This implies $x = 0$ and $p_o = -2$.

$$Q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1 x + \cdots = 1 + 4x^{-2}$$

This also implies $q_0 = 1$.

If that is true, then $m(m-1) + mp_0 + q_0 = m(m-1) - 2m + 1 = 0 \Rightarrow m = \frac{1}{2}\left(3 \pm \sqrt{5}\right) \neq \frac{1}{2}$.

REATTEMPT

I should have had the x, x^2 on these.

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + \cdots = -2x \;\; \Rightarrow \;\; p_0 = 0$$

$$x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1 x + \cdots = x^2(1 + 4x^{-2}) = x^2 + 4 \;\; \Rightarrow \;\; q_0 = 4$$

Now,

$$m(m-1) + mp_0 + q_0 = m(m-1) + 4 = 0 \;\Rightarrow\; x = \frac{1}{2} \pm \frac{1}{2}\sqrt{15}$$

REATTEMPT

I made a typo from the get-go!

$$4x^2 y'' - 8x^2 y' + (4x^2 + 1)y = 0 \;\; \Rightarrow \;\; y'' - 2y' + \frac{4x^2 + 1}{4x^2} y = 0$$

Thus,

$$P(x) = -2, \qquad Q(x) = 1 + \frac{1}{4} x^{-2}$$

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + \cdots = -2x \;\; \Rightarrow \;\; p_0 = 0$$

$$x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1 x + \cdots = x^2 \left(1 + \frac{1}{4} x^{-2}\right) = x^2 + \frac{1}{4} \;\; \Rightarrow \;\; q_0 = \frac{1}{4}.$$

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Then,

$$m(m-1)+mp_0+q_0 = m(m-1)+m(0)+\frac{1}{4}=0$$

Thus, $m_1 = m_2 = \frac{1}{2}$.

From here, we need to identify $\{a_n\}$.

We know that $a_0 \neq 0$. Let $a_0 = 1$

We know, $y_1 = x^{\frac{1}{2}}(a_0 + a_1x + a_2x^2 + \cdots)$. So, we need to find the summation by identifying $\{a_n\}$ and then, $y_2 = y_1 \log x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n$

[RP] For you guys, you can now try to solve the recursive relation for $\{a_n\}$.

We will return to this problem next week to finalize it. You all can try to finish it and post your response at PLEMacademy.com for this question to receive credit in the book.

[CORRECT GOING FORWARD ↓]

Formulaic Solution [After pondering and playing around, I concluded how to do this outside of the session and am solidifying the needed formulaic relation for exam (or homework) solutions. It should be respected that I know what I am doing and it still took 5-10 hours to solve. This is a Junior level ODE.]

$\mathcal{F}_1$ ***Formulae*** [Collect the proper formula, definitions, theorems, ...]

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2Q(x) = \sum_{n=0}^{\infty} q_n x^n, \qquad y'' + P(x)y' + Q(x)y = 0,$$

$$y_1 = x^{m_1}(a_0 + a_1x + a_2x^2 + \cdots), \qquad y_2 = y_1 \log x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n,$$

$$m(m-1)+mp_0+q_0 = 0.$$

To solve the one-solution Frobenius series problem, follow the steps provided.

1) Identify p_0 and q_0.
2) Solve for m.
3) Find the recursive relation.

$\mathcal{F}_2$ ***Formulate*** [Put the pieces together and solve the puzzle]

For p_0, q_0, put the ODE in the correct form—that is,

$$4x^2y'' - 8x^2y' + (4x^2+1)y = 0 \;\Rightarrow\; y'' + (-2)y' + \left(1 + \frac{1}{4}x^{-2}\right)y = 0.$$

Identify P, Q,

$$P(x) = -2, \qquad Q(x) = 1 + \frac{1}{4}x^{-2} \Rightarrow$$

Now, solve for p_0, q_0 using $xP(x)$ and $x^2Q(x)$.

$$xP(x) = -2, \qquad x^2Q(x) = 1 + \frac{1}{4}x^{-2}$$

$$\Rightarrow \; xP(x) = x(-2) = -2x, \qquad x^2Q(x) = x^2\left(1 + \frac{1}{4}x^{-2}\right) = x^2 + \frac{1}{4}$$

$$\Rightarrow \; xP(x) = \sum_{n=0}^{\infty} p_n x^n = -2x, \qquad x^2Q(x) = \sum_{n=0}^{\infty} q_n x^n = x^2 + \frac{1}{4}$$

$$\Rightarrow \; \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1x + p_2x^2 + \cdots = -2x, \qquad \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1x + q_2x^2 + \cdots = x^2 + \frac{1}{4}$$

$$\Rightarrow \; p_0 + p_1x + p_2x^2 + \cdots = -2x, \qquad q_0 + q_1x + q_2x^2 + \cdots = \frac{1}{4} + x^2$$

$$\Rightarrow \; p_0 + p_1x + p_2x^2 + \cdots = 0 + (-2)x + 0 + \cdots,$$

$$q_0 + q_1x + q_2x^2 + \cdots = \frac{1}{4} + (0)x + (1)x^2 + (0)x^3 + \cdots$$

Thus,

$$p_0 = 0, \qquad p_1 = -2, \qquad p_2 = 0 = p_3 = \cdots = p_k$$

$$q_0 = \frac{1}{4}, \qquad q_1 = 0, \qquad q_2 = 1, \qquad q_3 = 0 = q_4 = \cdots = q_k.$$

Now, solve for m,

$$m(m-1) + mp_0 + q_0 = m(m-1) + m(0) + \frac{1}{4} = 0 \; \Rightarrow \; m = \frac{1}{2} = m_1 = m_2.$$

Now that we have found m_1, m_2, we can find the recursive relation.

$$a_0 f(m) = 0$$

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$$a_1 f(m+1) + a_0(mp_1 + q_1) = 0$$

$$a_2 f(m+2) + a_0(mp_2 + q_2) + a_1[(m+1)p_1 + q_1] = 0$$

$$\vdots$$

$$f(m) = m(m-1) + mp_0 + q_0$$

We know $a_0 \neq 0$. Let $a_0 = 1$. We also know, $f(m) = m^2 - m + \frac{1}{4} = \left(m - \frac{1}{2}\right)^2$

Now, $p_1 = -2, q_1 = 0$

$$a_1 f(m+1) + a_0(mp_1 + q_1) = 0 \;\Rightarrow\; a_1\left[\left(\left(\left(\frac{1}{2}\right)+1\right) - \frac{1}{2}\right)^2\right] + (1)\left(\left(\frac{1}{2}\right)(-2) + 0\right) = 0$$

$$\Rightarrow\; a_1[(1)^2] + (1)\left(\left(\frac{1}{2}\right)(-2) + 0\right) = 0 \;\Rightarrow\; a_1 - 1 = 0 \;\Rightarrow\; a_1 = 1$$

Now, for a_2, $p_1 = -1,\ q_1 = 0,\ p_2 = 0, q_2 = 1$,

$$a_2 f(m+2) + a_0(mp_2 + q_2) + a_1[(m+1)p_1 + q_1] = 0$$

$$\Rightarrow\; a_2 f(m+2) + (1)\big(m(0) + (1)\big) + (1)[(m+1)(-2) + (0)] = 0$$

$$\Rightarrow\; a_2\left[\left((m+2) - \frac{1}{2}\right)^2\right] + (1)\big(m(0) + (1)\big) + (1)[(m+1)(-2) + (0)] = 0$$

$$\Rightarrow\; a_2\left[\left(\left(\frac{1}{2}+2\right) - \frac{1}{2}\right)^2\right] + (1)\big((1)\big) + (1)\left[\left(\frac{1}{2}+1\right)(-2)\right] = 0$$

$$\Rightarrow\; a_2\left[\left(\frac{5}{2} - \frac{1}{2}\right)^2\right] + 1 + (1)\left[\left(\frac{3}{2}\right)(-2)\right] = 0$$

$$\Rightarrow\; a_2[4] + 1 - 3 = 0 \;\Rightarrow\; 4a_2 = 2 \;\Rightarrow\; a_2 = \frac{1}{2}.$$

Now, we see that $a_0 = 1, a_1 = 1, a_2 = \frac{1}{2}$, … If we continue on, we will see the factorial relation—that is,

$$a_0 = \frac{1}{0!}, a_1 = \frac{1}{1!}, a_2 = \frac{1}{2!}, a_3 = \frac{1}{3!}, a_4 = \frac{1}{4!}, \dots$$

Now,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

Then,

$$y = c_1 y_1 + c_2 y_2 = c_1 \left(x^{m_1} \sum_{n=0}^{\infty} a_n x^n \right) + c_2 \left(y_1 \log x + x^{m_2} \sum_{n=0}^{\infty} a_n x^n \right)$$

$$= c_1 \left(x^{\frac{1}{2}} e^x \right) + c_2 \left(x^{\frac{1}{2}} e^x \log x + x^{\frac{1}{2}} e^x \right)$$

$$= c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x + c_2 x^{\frac{1}{2}} e^x$$

$$= (c_1 + c_2) x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x$$

$$= k_1 x^{\frac{1}{2}} e^x + k_2 x^{\frac{1}{2}} e^x \log x.$$

$\mathcal{F}_3$ ***Finalize*** [Clearly state the answer with appropriate citation/reference.]

Thus, by Frobenius 'one-solution' method, (we can go back to c as we are now exited the previous arithmetic).

$$y = c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x.$$

EXAM Method

[FIRST]

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n, \qquad y'' + P(x)y' + Q(x)y = 0,$$

$$y_1 = x^{m_1}(a_0 + a_1 x + a_2 x^2 + \cdots), \qquad y_2 = y_1 \log x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n,$$

$$m(m-1) + m p_0 + q_0 = 0.$$

[SECOND]

$$4x^2y'' - 8x^2y' + (4x^2 + 1)y = 0 \;\Rightarrow\; y'' + (-2)y' + \left(1 + \frac{1}{4}x^{-2}\right)y = 0.$$

Identify P, Q,

$$P(x) = -2, \qquad Q(x) = 1 + \frac{1}{4}x^{-2} \;\Rightarrow$$

Now, solve for p_0, q_0 using $xP(x)$ and $x^2Q(x)$.

$$xP(x) = -2, \qquad x^2Q(x) = 1 + \frac{1}{4}x^{-2}$$

$$\Rightarrow\; xP(x) = x(-2) = -2x, \qquad x^2Q(x) = x^2\left(1 + \frac{1}{4}x^{-2}\right) = x^2 + \frac{1}{4}$$

$$\Rightarrow\; xP(x) = \sum_{n=0}^{\infty} p_n x^n = -2x, \qquad x^2Q(x) = \sum_{n=0}^{\infty} q_n x^n = x^2 + \frac{1}{4}$$

$$\Rightarrow\; \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1x + p_2x^2 + \cdots = -2x, \qquad \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1x + q_2x^2 + \cdots = x^2 + \frac{1}{4}$$

$$\Rightarrow\; p_0 + p_1x + p_2x^2 + \cdots = -2x, \qquad q_0 + q_1x + q_2x^2 + \cdots = \frac{1}{4} + x^2$$

$$\Rightarrow\; p_0 + p_1x + p_2x^2 + \cdots = 0 + (-2)x + 0 + \cdots,$$

$$q_0 + q_1x + q_2x^2 + \cdots = \frac{1}{4} + (0)x + (1)x^2 + (0)x^3 + \cdots$$

Thus,

$$p_0 = 0, \qquad p_1 = -2, \qquad p_2 = 0 = p_3 = \cdots = p_k$$

$$q_0 = \frac{1}{4}, \qquad q_1 = 0, \qquad q_2 = 1, \qquad q_3 = 0 = q_4 = \cdots = q_k.$$

Now, solve for m,

$$m(m-1) + mp_0 + q_0 = m(m-1) + m(0) + \frac{1}{4} = 0 \;\Rightarrow\; m = \frac{1}{2} = m_1 = m_2.$$

Now that we have found m_1, m_2, we can find the recursive relation.

$$a_0 f(m) = 0$$

$$a_1 f(m+1) + a_0(mp_1 + q_1) = 0$$

$$a_2 f(m+2) + a_0(mp_2 + q_2) + a_1[(m+1)p_1 + q_1] = 0$$

$$\vdots$$

$$f(m) = m(m-1) + mp_0 + q_0$$

We know $a_0 \neq 0$. Let $a_0 = 1$. We also know, $f(m) = m^2 - m + \frac{1}{4} = \left(m - \frac{1}{2}\right)^2$

Now, $p_1 = -2, q_1 = 0$

$$a_1 f(m+1) + a_0(mp_1 + q_1) = 0 \;\Rightarrow\; a_1\left[\left(\left(\left(\frac{1}{2}\right)+1\right) - \frac{1}{2}\right)^2\right] + (1)\left(\left(\frac{1}{2}\right)(-2) + 0\right) = 0$$

$$\Rightarrow\; a_1[(1)^2] + (1)\left(\left(\frac{1}{2}\right)(-2) + 0\right) = 0 \;\Rightarrow\; a_1 - 1 = 0 \;\Rightarrow\; a_1 = 1$$

Now, for a_2, $p_1 = -1,\ q_1 = 0,\ p_2 = 0, q_2 = 1$,

$$a_2 f(m+2) + a_0(mp_2 + q_2) + a_1[(m+1)p_1 + q_1] = 0$$

$$\Rightarrow\; a_2 f(m+2) + (1)\big(m(0) + (1)\big) + (1)[(m+1)(-2) + (0)] = 0$$

$$\Rightarrow\; a_2\left[\left((m+2) - \frac{1}{2}\right)^2\right] + (1)\big(m(0) + (1)\big) + (1)[(m+1)(-2) + (0)] = 0$$

$$\Rightarrow\; a_2\left[\left(\left(\frac{1}{2}+2\right) - \frac{1}{2}\right)^2\right] + (1)\big((1)\big) + (1)\left[\left(\frac{1}{2}+1\right)(-2)\right] = 0$$

$$\Rightarrow\; a_2\left[\left(\frac{5}{2} - \frac{1}{2}\right)^2\right] + 1 + (1)\left[\left(\frac{3}{2}\right)(-2)\right] = 0$$

$$\Rightarrow\; a_2[4] + 1 - 3 = 0 \;\Rightarrow\; 4a_2 = 2 \;\Rightarrow\; a_2 = \frac{1}{2}.$$

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Now, we see that $a_0 = 1, a_1 = 1, a_2 = \frac{1}{2}$, … If we continue on, we will see the factorial relation—that is,

$$a_0 = \frac{1}{0!}, a_1 = \frac{1}{1!}, a_2 = \frac{1}{2!}, a_3 = \frac{1}{3!}, a_4 = \frac{1}{4!}, \dots$$

Now,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

Then,

$$y = c_1 y_1 + c_2 y_2 = c_1 \left(x^{m_1} \sum_{n=0}^{\infty} a_n x^n \right) + c_2 \left(y_1 \log x + x^{m_2} \sum_{n=0}^{\infty} a_n x^n \right)$$

$$= c_1 \left(x^{\frac{1}{2}} e^x \right) + c_2 \left(x^{\frac{1}{2}} e^x \log x + x^{\frac{1}{2}} e^x \right)$$

$$= c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x + c_2 x^{\frac{1}{2}} e^x$$

$$= (c_1 + c_2) x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x$$

$$= k_1 x^{\frac{1}{2}} e^x + k_2 x^{\frac{1}{2}} e^x \log x.$$

[THIRD]

Thus, by Frobenius 'one-solution' method,

$$y = c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x.$$

[FC] How do we know this answer satisfies the initial ODE? Because, we could have coincidentally gotten the same answer as the author(s) solution.

We have to take the derivatives of the answer and plug into the original equation to see if we get $0 = 0$.

$$y = c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x, \qquad y' = \frac{d}{dx}\left[c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x \right]$$

$$y'' = \frac{d^2}{dx^2}\left[c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x \right]$$

Then,

$$4x^2y'' - 8x^2y' + (4x^2 + 1)y = 0$$

$$\Rightarrow \quad 4x^2 \frac{d^2}{dx^2}\left[c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x\right]$$

$$-8x^2 \frac{d}{dx}\left[c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x\right]$$

$$+(4x^2 + 1)\left[c_1 x^{\frac{1}{2}} e^x + c_2 x^{\frac{1}{2}} e^x \log x\right] = 0.$$

We can go through the motions to show that this reduces to $0 = 0$. However, technology is a good use for this. ChatGPT agrees that the final result is $0 = 0$. For your own learning experience, you should show that the left side does reduce to zero, which is just a bunch of arithmetic.

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Textbooks used as Reference

[1] Larson, Ron, and Robert P. Hostetler. Precalculus: Real Mathematics, Real People, 7th ed. Boston: Cengage Learning. This book was found as a PDF via:

https://resources.finalsite.net/images/v1761258636/ppsnet/aopoxfaokrwyzs0alpug/precalculus_7th_-_larson_hostetler_2.pdf

---

[2] Barnett, R., Ziegler, M., Byleen, K., & Sobecki, D. (2015). College Algebra with Trigonometry (9th ed.). McGraw-Hill Education. This book was found as a PDF via:

https://marinazahara22.wordpress.com/wp-content/uploads/2013/10/college_algebra_with_trigonometry.pdf

---

[3] Young, H. D., & Freedman, R. A. (2015). Sears and Zemansky's University Physics with Modern Physics (13th ed.). Pearson. This book was found as a PDF via:

https://www.andrew.cmu.edu/user/bbji/Desktop/University%20Physics%20with%20Modern%20Physics%2015th%20Edition%20By%20Hugh%20D.%20Young_compressed.pdf

---

[4] Halliday, D., Resnick, R., & Walker, J. (2014). *Fundamentals of Physics* (10th ed., extended). Wiley.

https://dl.icdst.org/pdfs/files4/39e1817b05cf155433309dbb2f3289fe.pdf

---

[5] Lay, D. C., Lay, S. R., & McDonald, J. J. (2016). *Linear algebra and its applications* (5th ed.). Pearson.

https://github.com/vzardlloo/trans/blob/master/origin/Linear%20Algebra%20and%20Its%20Applications%205th%20Edition.pdf

---

[6] Strang, G. (2006). *Linear algebra and its applications* (4th ed.). Brooks/Cole.

https://rksmvv.ac.in/wp-content/uploads/2021/04/Gilbert_Strang_Linear_Algebra_and_Its_Applicatio_230928_225121.pdf

---

[7] Thomas, G. B., Weir, M. D., Hass, J., & Heil, C. (2018). Thomas' calculus: Early transcendentals (13th ed.). Pearson.

https://www.rkmmanr.org/uploads/Thomas_Calculus.pdf

---

[8] Griffiths, D. J., & Schroeter, D. F. (2018). Introduction to quantum mechanics (3rd ed.). Cambridge University Press.

https://www.fisica.net/mecanica-quantica/Griffiths%20-%20Introduction%20to%20quantum%20mechanics.pdf

---

[9] Bird, J. (2014). Engineering mathematics (5th ed.). Routledge.

https://math24.wordpress.com/wp-content/uploads/2016/02/engineering-mathematics.pdf

---

[10] Bird, J. (2014). Higher engineering mathematics (5th ed.). Routledge.

https://uodiyala.edu.iq/uploads/PDF%20ELIBRARY%20UODIYALA/EL24/HigherEM.pdf

---

[11] Zill, D. G., & Cullen, M. R. (2009). Differential equations with boundary-value problems (7th ed.). Brooks/Cole Cengage Learning.

http://www.uop.edu.pk/ocontents/DIFFERENTIAL%20EQUATIONS%20with%20Boundary-Value%20Problemsa%20Zill%20Cullen(1).pdf

---

[12] Boas, M. L. (2006). Mathematical methods in the physical sciences (3rd ed.). John Wiley & Sons.

https://www.christs.cam.ac.uk/sites/default/files/inline-files/0a187866618ca3049030ec5014860ae8-original.pdf

---

[13] Ryan, S. D. (2022). *Mathematical methods for the physical sciences: Two semester course*.

https://academic.csuohio.edu/shawn-ryan/wp-content/uploads/sites/66/2022/06/MathMethodsBook-compressed.pdf

[14] Hibbeler, R. C. (2009). Engineering mechanics: Statics (12th ed.). Pearson Prentice Hall.

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https://gessknust.wordpress.com/wp-content/uploads/2015/09/engineering-mechanics-statics-r-c-hibbeler-12th-edition.pdf

[15] Hibbeler, R. C. (2016). Engineering mechanics: Statics (14th ed.). Pearson.

https://api.pageplace.de/preview/DT0400.9781292088785_A26856513/preview-9781292088785_A26856513.pdf

[16] Kreyszig, E. (2011). *Advanced engineering mathematics* (10th ed.). Wiley.

https://drspmaths.wordpress.com/wp-content/uploads/2019/01/advanced-engineering-mathematics_by-erwin-kreyszig-__10th_edition-1.pdf

[17] Calculus: Early Transcendentals (7th Edition) — James Stewart. (2012). Calculus: Early transcendentals (7th ed.). Brooks/Cole Cengage Learning.

https://elcom-team.com/Subjects/calculus%202/%D8%A7%D9%84%D9%83%D8%AA%D8%AA%D8%A8%20%D9%88%D8%A7%D9%84%D8%AD%D9%84%D9%88%D9%84/calc2-book(7ed).pdf

[18] Anton, H., Bivens, I. C., & Davis, S. (2012). Calculus (10th ed.). John Wiley & Sons.

https://3lihandam69.wordpress.com/wp-content/uploads/2018/10/calculus-10th-edition-anton.pdf

[19] Sullivan, M. (2011). Algebra and trigonometry (9th ed.). Pearson.

https://home.ufam.edu.br/andersonlfc/Nivelamento_Matem%C3%A1tica/Algebra%20&%20Trigonometry%20-%20Sullivan/Sullivan%20Algebra%20&%20Trigonometry%209th%20txtbk.pdf

[20] Rosen, K. H. (2011). *Elementary number theory and its applications* (6th ed.). Pearson.

https://www.researchgate.net/profile/Ajith-Nayanananda/post/I_want_to_ask_what_list_do_you_suggest_with_priority_from_beginner_to_professional_level_to_fully_learn_algebraic_analytic_number_theory/attachment/5f86e95ce66b860001a89206/AS%3A946510439145473%401602677082663/download/Rosen-Number-Theory-6ed.pdf

[21] Boyce, W. E., DiPrima, R. C., & Meade, D. B. (2017). Elementary differential equations and boundary value problems (11th ed.). John Wiley & Sons.

https://scienceclopedia.wordpress.com/wp-content/uploads/2019/09/william-e.-boyceelementary-differential-equations-and-boundary-value-problems-wiley-2017.pdf

[22] Kreyszig, E. (2011). Advanced engineering mathematics (10th ed.). John Wiley & Sons.ce

http://www.uop.edu.pk/ocontents/AdvancedEngineeringMathematics.pdf

[23] Goodwine, B. (2018). Engineering differential equations: Theory and applications. Springer.

http://bionics.seas.ucla.edu/education/MAE_182A/Book_Eng_differentia_equation.pdf

[24] Gallian, J. A. (2010). *Contemporary abstract algebra* (7th ed.). Brooks/Cole, Cengage Learning.

https://www.ayman-badawi.com/2019%20%20Contemporary+Abstract+Algebra-7.pdf

[25]

https://nucleares.unam.mx/~martinel/griffiths_4ed.pdf

[26] Author Last Name, First Initial. (Year). Title or description of image [Image]. Source or platform.

https://3lihandam69.wordpress.com/wp-content/uploads/2018/10/calculus-10th-edition-anton.pdf

[27] marsden

[28] Stewart, J., Redlin, L., & Watson, S. (2016). Precalculus: Mathematics for calculus (7th ed.). Cengage Learning.

https://jhevonorg.wordpress.com/wp-content/uploads/2019/09/precalculus_mathematics_for_calculus_7th-first-five-sections.pdf

[29] Serway, R. A., Vuille, C., & Faughn, J. S. (2018). College physics (11th ed.). Cengage Learning.

https://fizikamentor.wordpress.com/wp-content/uploads/2018/04/college-physics.pdf

[30] Dolciani, M. P., Berman, S. L., & Freilich, J. (1962). Modern algebra: Structure and method. Book 1. Houghton Mifflin.

https://ia801400.us.archive.org/26/items/modernalgebrastr01dolc_2/modernalgebrastr01dolc_2.pdf

© 2026 | FalseRealityPublishing.com | False Reality Publishing LLC

[31] Nagy, Gabriel. Ordinary Differential Equations. Mathematics Department, Michigan State University, January 18, 2021. Available at: Michigan State University ODE PDF

https://users.math.msu.edu/users/gnagy/teaching/ode.pdf

[32] Simmons, George F. Differential Equations with Applications and Historical Notes. 3rd ed. Boca Raton, FL: CRC Press, 2016. Available at: Differential Equations PDF

https://horizons-2000.org/92.%20Misc%20Files/Reading/Differential%20Equations--George%20Simpson.pdf

[33] Olver, P. J. (2014). Introduction to partial differential equations. Springer. https://doi.org/10.1007/978-3-319-02099-0

https://weblibrary.mila.edu.my/upload/ebook/engineering/2014_Book_IntroductionToPartialDifferent.pdf

[34] Salsa, S. (2008). *Partial differential equations in action: From modelling to theory*. Springer.

https://www.sgo.fi/~j/baylie/Partial%20Differential%20Equations%20in%20Action%20-%20From%20Modelling%20to%20Theory%20-%20S.%20Salsa%20(Springer,%202008)%20WW.pdf

[35] Logan, J. D. (2015). *Applied partial differential equations* (3rd ed.). Springer.

https://weblibrary.mila.edu.my/upload/ebook/engineering/2015_Book_AppliedPartialDifferentialEqua.pdf

[36] Stewart, J. (2016). *Calculus: Early Transcendentals* (8th ed.). Cengage Learning.

https://dzackgarza.com/assets/books/Stewart.pdf

[37] Anton, H., & Rorres, C. (2005). *Elementary linear algebra: Applications version* (10th ed.). John Wiley & Sons.

https://dusolution.wordpress.com/wp-content/uploads/2015/06/elementary_linear_algebra_10th_edition.pdf

---

[38] Anton, H., & Kaul, A. (2021). *Elementary linear algebra* (12th ed.). John Wiley & Sons.

https://www.studyhalo.com/media/resources/resources/MAT1503/Textbook/MAT1503_-_Prescribed_book.pdf

---

[39] Nagle, R. K., Saff, E. B., & Snider, A. D. (2018). *Fundamentals of differential equations* (9th ed.). Pearson.

https://faculty.ksu.edu.sa/sites/default/files/R.-Kent-Nagle-Edward-B.-Saff-Arthur-David-Snider-Fundamentals-of-Differential-Equations-Pearson-2017.pdf

---

[40] Anderson, J. D., Jr. (2011). Fundamentals of aerodynamics (5th ed.). McGraw-Hill.

https://www.rexresearch1.com/AeroEngineeringLibrary/FundamentalsAerodynamics.pdf

[SM.40] Anderson, J. D., Jr. *Instructor's solutions manual to accompany Fundamentals of aerodynamics* (4th ed.). McGraw-Hill.

https://dn790002.ca.archive.org/0/items/56611831FundamentalsOfAerodynamics4eSolutionManual1/56611831-Fundamentals-of-Aerodynamics-4e-Solution-Manual%281%29.pdf

---

[41] Chapra, S. C., & Canale, R. P. (2010). Numerical methods for engineers (6th ed.). McGraw-Hill.

https://www.mbit.edu.in/wp-content/uploads/2020/05/Numerical_methods_for_engineers_for_engi.pdf

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