Diagonalization: Eigenvalues and Eigenvectors

[26] Lipschutz, S., & Lipson, M. (2018). Schaum's outline of linear algebra (6th ed.). McGraw-Hill Education.

https://cs.mcgill.ca/~akroit/math/algebra/Schaum%20Outline%20of%20Linear%20Algebra.pdf

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NOTATION REVIEW

I would like to address $A = [a_{ij}]$ and what that means. If A is $m \times n$ ($T: \mathbb{R}^n \to \mathbb{R}^m \equiv A_{m\times n}$)

$$A = [a_{ij}] \equiv \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = A_{m\times n}$$

In this book, ($T: \mathbb{R}^n \to \mathbb{R}^m \equiv A_{m\times n}$) would be $T: \mathbf{R}^n \to \mathbf{R}^m$.

In other books, **R** means something completely different. Hence, you must stick with the book assigned else the algorithm is incorrect even if it produces the matching answer.

NOTE The student requesting this textbook review and lesson guide also has professor notes. The professors—if different from the book—trumps the book's format, notation, and solution—that is, if the professor uses a different class of notation and a different solution than the book's proposed method, you must follow the professor's method. This is why lecture attendance is required. Professors cannot legally force adults to be in class. Meaning-*attendance is still required even though the syllabus may say it is not*—you are an adult.

This symbol and symbols like it often redefine axiomatic properties under vector space rules. $\otimes$. Example, $a \otimes b = (a)(b) - 4$. Let $a = 3$ and $b = 1$. The $3 \otimes 1 = (3)(1) - 4 = 3 - 4 = -1$. Traditional multiplication: $(3) \times (1) \equiv (3)(1) = 3$.

QUESTION 9.32

Find a matrix A whose minimal polynomial is $f(t) = t^3 - 8t^2 + 5t + 7$.

Simply let $A = \begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & -5 \\ 0 & 1 & 8 \end{bmatrix}$, the companion matrix of $f(t)$ [defined in Example 9.12(b)].

ANSWER(SOLUTION)

No answer or solution provided.

$\mathcal{F}_1$ [Formulae] 50% of Exam Grade

NOTE I took a screenshot but you as a student should type this out for memory.

THEOREM 9.15: The minimal polynomial $m(t)$ of a matrix (linear operator) A divides every polynomial that has A as a zero. In particular, $m(t)$ divides the characteristic polynomial $\Delta(t)$ of A.

Image found on Pg. 305 for Educational Purposes

IMPORTANT In the reading, the author works start to finish. The question we are doing is going in reverse. The posed technique is to get experience reverse engineering problems.

(b) Consider an arbitrary monic polynomial:

$$f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

Let $C(f)$ be the n-square matrix with 1's on the *subdiagonal* (consisting of the entries below the diagonal entries), the negatives of the coefficients in the last column, and 0's elsewhere as follows:

$$C(f) = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}$$

Then $C(f)$ is called the *companion matrix* of the polynomial $f(t)$. Moreover, the minimal polynomial $m(t)$ and the characteristic polynomial $\Delta(t)$ of the companion matrix $C(f)$ are both equal to the original polynomial $f(t)$.

Image found on Pg. 306 for Educational Purposes

The matrix provided is built with the above method.

$$A = \begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & -5 \\ 0 & 1 & 8 \end{bmatrix}, \qquad f(t) = (1)t^3 - 8t^2 + 5t + 7 = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

Now we know how to find the companion matrix. [BE CAREFUL] The leading term has coefficient 1. On an exam, you may not have that—meaning, this method would be the wrong method. E.g., $f(t) = 5t^3 - 8t - t - 1 \leftrightarrow 5 \neq 1$.

• Before going forward, after reviewing the theorems and definitions, I see that we have the **minimal polynomial** which divides the **characteristic polynomial** which is found by subtracting the identity matrix with λ from A and taking the determinant—that is, the same process for finding Eigenvalues.

[VERY IMPORTANT] There are no solutions or answers for this question. Your professor may have a trick to solve this that is not in the book. You will only learn this from their notes. *Anything from any other source is unacceptable*.

$\mathcal{F}_2$ [Formulate] 50% of Exam Grade

Now, we have a 3×3 matrix. The minimal polynomial contains a cube, which implies that the minimal polynomial is in fact the characteristic polynomial. In the book's example on page 306, the minimal polynomial of the characteristic polynomial is of order 2; ours is of order 3.

At this time, I believe that $f(t) = m(t) = t^3 - 8t^2 + 5t + 7 \equiv \Delta(t)$. If that is the case, then

$$\det(A - \lambda I) = t^3 - 8t^2 + 5t + 7.$$

The theorem states, "The characteristic polynomial of the companion matrix is the same as the original matrix."

$$\det\left(\begin{bmatrix}0 & 0 & -7\\ 1 & 0 & -5\\ 0 & 1 & 8\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = \det\left(\begin{bmatrix}0 & 0 & -7\\ 1 & 0 & -5\\ 0 & 1 & 8\end{bmatrix} - \begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\right) = \begin{vmatrix}-\lambda & 0 & -7\\ 1 & -\lambda & -5\\ 0 & 1 & 8-\lambda\end{vmatrix}$$

$$\begin{vmatrix}-\lambda & 0 & -7\\ 1 & -\lambda & -5\\ 0 & 1 & 8-\lambda\end{vmatrix} = +(-\lambda)\begin{vmatrix}-\lambda & -5\\ 1 & 8-\lambda\end{vmatrix} - (0)|\dots| + (-7)\begin{vmatrix}1 & -\lambda\\ 0 & 1\end{vmatrix}$$

$$= +(-\lambda)[(-\lambda)(8-\lambda) - (-5)(1)] - (0)|\dots| + (-7)[(1)(1) - (-\lambda)(0)]$$

$$= -\lambda[-8\lambda + \lambda^2 + 5] - 7 = -\lambda^3 + \lambda^2 8 - \lambda 5 - 7$$

MISTAKE ↑ I chose the formula, $\det(A - \lambda I)$ which is conflicts with this theorem.

What I just did was leave the book. The book (page 296), uses formula

$$\Delta(t) = \det(I_n t - A) = (-1)^n \det(A - I_n t).$$

NOTE The formula I used, when set to zero, still provides the same eigenvalues but not the same characteristic polynomial. Simply fix. $n = 3$ for $3 \times 3 = n \times n$ matrix.

$$(-1)^3 \det\left(\begin{bmatrix}0 & 0 & -7\\ 1 & 0 & -5\\ 0 & 1 & 8\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = (-1)^3(-\lambda^3 + \lambda^2 8 - \lambda 5 - 7)$$

$$= t^3 - 8t^2 + 5t + 7 = m(t).$$

We have shown that the companion matrix has the same characteristic polynomial as the original matrix, A that we seek. That being said, is the companion matrix equal to A?

They gave us the minimal matrix. We found the characteristic polynomial of the companion matrix. It turns out that the minimal polynomial is equal to the characteristic polynomial of the companion matrix and by theorem, the characteristic polynomial of the companion

matrix is, in fact, the same as the characteristic polynomial of the original matrix. Thus, leading me to believe that

$$C(f) = \begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & -5 \\ 0 & 1 & 8 \end{bmatrix} = A.$$

Now, I am not 100% positive on this. I can sift through the book some more, but I do wish to test the waters with AI but not for a solution. I will address ChatGPT to provide an answer but no solution to cross my thinking. If you are really hardcore, you can find a way to prove your answer under the full circle of math. At this level for a beginner in engineering math, we would look to software for assistance. Mathematicians and physicists would do the analysis rigorously.

$\mathcal{F}_3$ [Finalize] 0% of Exam Grade

We asked ChatGPT, "I'm going to give you a question from the Schwam's linear algebra book with, whoever, the one you just cited in the other chat room. What I want you to do is just give me the answer to the question. I do not want the solution. Just give me straight up the answer to the question, no solution, nothing else, okay?"

And yes! She concurs that our logic is correct.

Thus, by theorem and definition according to the text, we conclude that

$$A = \begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & -5 \\ 0 & 1 & 8 \end{bmatrix}.$$

EXAM(HW) SOLUTION (do by hand if proctored exam is handwritten)

We know that $m(t)$ divides $\Delta(t)$ and that $\Delta(t)$ is the same as the characteristic polynomial of the companion matrix.

Since the highest degree of $m(t)$ is 3 and we are working with a 3×3 matrix, we see that

$$m(t) \equiv \Delta(t).$$

Hence, our companion matrix, is A. **Thus, by definition and theorem, 9.15 & 9.16**,

$$A = C(f) = \begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & -5 \\ 0 & 1 & 8 \end{bmatrix}.$$

Double check with professor that this is enough for full credit.

NOTE For exams on such a question, I would prepare for a minimal equation that is of lower order than the characteristic polynomial—i.e., work the question in the reading, 9.12(b) as that will probably be the question word-for-word – number-for-number on the exam!

[FULL CIRCLE] Can you prove that the answer satisfies the question? HINT Show $m(t)$ divides $\Delta(t)$.