Exact Equations in Differential Equations

QUESTION [21] 2.6.7 Determine whether the equation is exact. If it is exact, find the solution.

$$\left(\frac{y}{x} + 6x\right) + (\ln x - 2)y' = 0, \qquad x > 0.$$

ANSWER

$$y \ln x + 3x^2 - 2y = c.$$

Before we begin to solve this question, we should acknowledge the condition, $x > 0$. What does this mean? Why is it stated?

Because the logarithmic function $f(x) = \log x$ is defined to be only valid for values $x > 0$, the condition is stated.

NOTE $\log x \equiv \log_e x \equiv \ln x \equiv \frac{\ln x}{\ln e}$. Change of base $\log_b a = \frac{\ln a}{\ln b}$. $\log x \equiv \ln x$. Base 10 is a TI-84 thing. $\log x$ and $\ln x$ are the same. $\log x$ is base e. For $\log x$ or $\ln x$, the logarithmic function is defined to be $x > 0$ for all x. Thus, the acknowledgment of the restriction.

For differential equations, it is important to first cite the theorem(s)/definition(s) prior to attempting to solve.

EXACT EQUATION Theorem 2.6.1

$$M(x, y) + N(x, y)y' = 0$$

$$\text{Exact iff } M_y = N_y \text{ iff } \psi_x = M,\ \psi_y = N$$

$$\text{Also, } \psi_{xy}, \psi_{yx} \text{ are continuous.}$$

The verbatim theorem above is what we will work from. It is good to recognize that this is the same exact problem as solving conservative vector function in multivariable calculus.

Step 1 for Differential Equations: Identify the ODE,

$$\left(\frac{y}{x} + 6x\right) + (\ln x - 2)y' = 0, \qquad x > 0.$$

The above equation with dependent variable, y and independent variable x says that the ODE is first-order-linear-nonhomogeneous. We can put it into exact form from different books as well. We are in [21].

$\mathcal{F}_1$ Identify the ODE “first-order-linear-nonhomogeneous” with potential exactness.

Next, identify the method needed to solve the ODE. In this case, we told what to do. However, that may not be the case on exams. You should take account for formation to be prepped for exam alteration of structure.

$$M(x,y) = \frac{y}{x} + 6x, \qquad N(x,y) = (\ln x - 2).$$

$\mathcal{F}_2$ Formulation of the theorem(s)/definition(s)/equation(s)

The equation is exact iff $M_y = N_y$.

$$M_y = \frac{\partial}{\partial y}\left(\frac{y}{x} + 6x\right) = \frac{\partial}{\partial y}\frac{y}{x} + \frac{\partial}{\partial y}6x = \frac{1}{x}\frac{\partial}{\partial y}y + 6x\frac{\partial}{\partial y}(1) = \frac{1}{x}\frac{d}{dy}y + 6x\frac{d}{dx}(1)$$

$$= \frac{1}{x}\frac{dy}{dy} + 6x(0) = \frac{1}{x}(1) + (0) = \frac{1}{x}.$$

$$N_y = \frac{\partial}{\partial x}(\ln x - 2) = \frac{\partial}{\partial x}\ln x - \frac{\partial}{\partial x}2 = \frac{d}{dx}\ln x - (0) = \frac{1}{x}\frac{d}{dx}x = \frac{1}{x}\frac{dx}{dx} = \frac{1}{x}.$$

Thus,

$$M_y = \frac{1}{x} = \frac{1}{x} = N_x.$$

We can now see by the theorem that the equation is indeed exact.

So, how do we solve it? Well, this is the same as a conservative function in many variable calculus techniques. But, we are not in calculus-MV, so we must stick with the book!!!

Let $\psi_x = M$ and $\psi_y = N$

$$\psi_x = M(x,y) = \frac{y}{x} + 6x, \qquad \psi_y = N(x,y) = \ln x - 2.$$

Now, we have a relation of antiderivative nature. Let us take the antiderivatives.

$$\psi_x = \frac{y}{x} + 6x, \qquad \psi_y = \ln x - 2 \quad \Rightarrow \quad \psi = \int \frac{y}{x} + 6x\, \partial x, \qquad \psi = \int \ln x - 2 \ \partial y$$

$$\Rightarrow \quad \psi = \int \frac{y}{x} + 6x\, dx, \qquad \psi = \int \ln x - 2 \ dy$$

$$\Rightarrow \;\psi = \int \frac{y}{x}dx + \int 6x\,dx, \qquad \psi = \int \ln x\,dy - \int 2\;dy$$

$$\Rightarrow \;\psi = y\int \frac{1}{x}dx + 6\int x\,dx, \qquad \psi = \ln x\int dy - 2\int dy$$

$$\Rightarrow \;\psi = y\ln|x| + 6\left[\frac{1}{2}x^2\right] + g_1(y), \qquad \psi = \ln x\,[y] - 2[y] + g_2(x)$$

We now have two equation that are the same but with mix and match variable.

$$\psi = y\ln|x| + 3x^2 + g_1(y), \qquad \psi = y\ln x - 2y + g_2(x)$$

$$\Rightarrow \;\psi = y\ln|x| + 3x^2 + g_1(y), \qquad \psi = y\ln x + (-2y) + g_2(x)$$

$$\Rightarrow \;\psi = y\ln|x| + 3x^2 + g_1(y), \qquad \psi = y\ln x + (-2y) + g_2(x)$$

$$h(x,y) = y\ln x.$$

Thus, you can omit the absolute value since the original statement states $x > 0$, we know that it is positive so we can drop the absolute value. (Don't' forget the scalar constant.)

$$\therefore \psi = y\ln x + 3x^2 - 2y + C.$$

NOTE I have written the answer as $\psi = y\ln x + 3x^2 - 2y + C$, however, the books answer is

$$y\ln x + 3x^2 - 2y = c.$$

Do you know why?

$\mathcal{F}_3$ Finalize the answer stated correctly.

In a 3D calculus situation, it is common to add the constant and equate to a function.

$$y\ln x + 3x^2 - 2y = c \quad vs \quad \psi = y\ln x + 3x^2 - 2y + C$$

The former is the same relation—that is, one is state as a function that equals zero whereas the other is stated as a function which equals zero.

$$\psi = y\ln x + 3x^2 - 2y + C = 0, \qquad y\ln x + 3x^2 - 2y - c = 0.$$

NOTE A constant with a constant is a constant. $-c$ is a variation of C. $\psi = 0$ in this case. Depending on the course you are in, you will choose to label. We are master of labeling.

TEST RESPONSE

QUESTION Solve the ODE
$$\left(\frac{y}{x} + 6x\right) + (\ln x - 2)y' = 0, \qquad x > 0.$$

(Step 1) By exact equation $M + Ny' = 0$ if $M_y = N_x$, then it is exact.

(Step 2)
$$\psi_x = M(x,y) = \frac{y}{x} + 6x, \qquad \psi_y = N(x,y) = \ln x - 2.$$

$$\Rightarrow \ \psi = \int \frac{y}{x} + 6x\,dx, \qquad \psi = \int \ln x - 2\ dy$$
$$\Rightarrow \ \psi = \int \frac{y}{x}dx + \int 6x\,dx, \qquad \psi = \int \ln x\,dy - \int 2\ dy$$
$$\Rightarrow \ \psi = y\int \frac{1}{x}dx + 6\int x\,dx, \qquad \psi = \ln x \int dy - 2\int dy$$
$$\Rightarrow \ \psi = y\ln|x| + 6\left[\frac{1}{2}x^2\right] + g_1(y), \qquad \psi = \ln x\,[y] - 2[y] + g_2(x)$$

$$\Rightarrow \ \psi = y\ln|x| + 3x^2 + g_1(y), \qquad \psi = y\ln x - 2y + g_2(x)$$

$$\Rightarrow \ \psi = y\ln|x| + 3x^2 + g_1(y), \qquad \psi = y\ln x + (-2y) + g_2(x)$$

$$\Rightarrow \ \psi = y\ln|x| + 3x^2 + g_1(y), \qquad \psi = y\ln x + (-2y) + g_2(x).$$

$\mathcal{F}_3$ Finalize and state the answer with citation.

By "The Exact Theorem"
$$y\ln x + 3x^2 - 2y = c.$$

[FC] How do we know this is right? The book matches but how do we know?

It is now your turn to show the answer satisfies the question.

